Stochastic LP with Recourse

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Examples

- Water Allocation
- Production Planning
- Transportation Problem (random demand)
- 2-Stage Stochastic Programming

EXAMPLE

Water Resources Planning Under Uncertainty

A water system manager must allocate water from a stream to three users:

- municipality
- industrial concern
- agricultural sector

Use	Request	Net Benefit per unit
1. Municipality	2	100
2. Industrial	3	50
3. Agricultural	5	30

Let X_i = amount of water allocated to use #i

The optimal allocation might be found by solving the LP:

$$\begin{aligned} \text{Max } & 100X_1 + 50X_2 + 30X_3 \\ \text{subject to } & X_1 + X_2 + X_3 \le Q \\ & 0 \le X_1 \le 2 \\ & 0 \le X_2 \le 3 \\ & 0 \le X_3 \le 5 \end{aligned}$$

But the decision must be made before the quantity Q of the available water is known!

$$\begin{aligned} \text{Max } & 100X_1 + 50X_2 + 30X_3 \\ \text{subject to } & X_1 + X_2 + X_3 \le Q \\ & 0 \le X_1 \le 2 \\ & 0 \le X_2 \le 3 \\ & 0 \le X_3 \le 5 \end{aligned}$$

Random variable with known probability distribution

How should the water be allocated before the quantity available is known?

Streamflow Distribution

i	q_i	$P\{Q=q_i\}$
1	4	20%
2	10	60%
3	17	20%

Use	Request	Loss per unit shortfall
1. Municipality	2	250
2. Industrial	3	75
3. Agricultural	5	60

If more water is promised than can be later delivered, then a loss results from the need either to acquire alternative sources &/or to reduce consumption.

What is the "optimal" quantity to allocate to each use, if Q is not yet known?

solution

EXAMPLE

Production Planning with Uncertain Resources

Par, Inc., a manufacturer of golf bags, must schedule production for the next quarter.

PRODUCTION TIME/BAG IN EACH DEPARTMENT

product	Cutting & Dyeing	Sewing	Finishing	Inspect Package
Standard	$\frac{7}{10}$ hr	$\frac{1}{2}$ hr	1 hr	$\frac{1}{10}$ hr
Deluxe	1 hr	$\frac{5}{6}$ hr	$\frac{2}{3}$ hr	$\frac{1}{4}$ hr

The company can sell as many bags as can be produced at a profit of $10 per standard bag and $9 per deluxe bag.

$$\begin{aligned}\text{Max }\; & 10X_1 + 9X_2\\ \text{subject to }\; & \tfrac{7}{10}X_1 + X_2 \le 630\\ & \tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 \le 600\\ & X_1 + \tfrac{2}{3}X_2 \le 708\\ & \tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 \le 135\\ & X_1 \ge 0,\; X_2 \ge 0\end{aligned}$$

Dept.	Available hrs.
C&D	630
SEW	600
FIN	708
I&P	135

Based upon current commitments, the hours available in each department for the next quarter are computed.

However, the firm has submitted bids on two contracts, which if successful would reduce the hours available for producing golf bags.

Contract	probability	Production Hours Reqd C&D	SEW	FIN	I&P
#1	50%	50	40	80	10
#2	40%	30	50	70	15

A production schedule for standard & deluxe bags must be chosen before learning which contracts, if any, were awarded to the firm. Afterwards, the production schedule may be modified somewhat, but extra costs are incurred in doing so...

For each scenario, we compute the available hours in each department (subtracting the hours used to fill any contracts which are won)

Available hrs.

Dept.	scenario #0	#1	#2	#3
C&D	630	580	600	550
SEW	600	560	550	510
FIN	708	628	638	558
I&P	135	125	120	110

Recourses

- Scheduling overtime in C&D at \$5/hr
  SEW at \$6/hr
  FIN at \$8/hr
  I&P at \$4/hr
  (only 100 hrs OT available in FIN)
- Schedule additional production of standard bags, at a reduced profit of \$8/bag

solution

Stochastic LP with Recourse

Linear Constraints

$Ax + By = b$
$x \ge 0,\; y \ge 0$

Sequence of Events

- x is selected by the decision-maker
- the random variable b is observed
- the decision-maker must choose y so as to satisfy constraint, i.e.
  $By = b - Ax$

Costs Incurred $cx + dy$

Second-Stage Problem

$$\phi(x,b) = \text{Minimum } dy \quad \text{s.t. } By = b - Ax,\; y \ge 0$$

both x & b are fixed

Since b is a random variable, so also is $\phi(x,b)$ for fixed x.

First-Stage Problem

$$\text{Minimize } cx + E_b[\phi(x,b)] \quad \text{subject to } \phi(x,b) < \infty$$

i.e., 2nd-Stage Problem should be feasible for all possible values of b

Minimize the sum of the first-stage cost and the expected cost of the 2nd stage:

$E_b[\phi(x,b)]$ is the expected cost of the second stage, for fixed x
This is generally a nonlinear, but convex, function of x.

Discrete RHS distribution

Suppose that the right-hand-side vector b is "drawn" from a finite set of possible RHSs $\{b^1, b^2, \dots, b^k\}$ with probabilities $p_1, p_2, \dots p_k$.

Define a second-stage (recourse) vector for each of the possible RHSs: $y^1, y^2, \dots y^k$

Then the recourses must be selected so that given the first-stage decision x, this system of equations is satisfied:

$$\begin{cases} Ax + By^1 = b^1 \\ Ax + By^2 = b^2 \\ \vdots \quad \vdots \quad \vdots \\ Ax + By^k = b^k \end{cases}$$

$$\underbrace{p_i}_{\text{probability of outcome } b^i} \; \underbrace{dy^i}_{\text{cost if recourse } y^i \text{ is selected}}$$

The expected value of the second-stage cost is

$$\boxed{p_1\, dy^1 + p_2\, dy^2 + \dots + p_k\, dy^k}$$

$$\begin{array}{ll} \text{Minimize } cx + p_1 dy^1 + p_2 dy^2 + \dots + p_k dy^k & \leftarrow \text{First-stage cost } \textit{plus} \text{ expected } 2^{nd}\text{-stage cost} \\ \text{subject to } \begin{array}{lllll} Ax + By^1 & & & & = b^1 \\ Ax & + By^2 & & & = b^2 \\ Ax & & + By^3 & & = b^3 \\ \vdots & & & \ddots & \vdots \\ Ax & & & + By^k & = b^k \end{array} & \\ \quad x \geq 0, y^1 \geq 0, \dots, y^k \geq 0 & \end{array}$$

Notice the block-angular structure of the coefficient matrix....

Question: Could the Dantzig-Wolfe decomposition technique be used in order to decompose this problem into smaller subproblems?

$$\begin{array}{l} \text{Minimize } cx + p_1 dy^1 + p_2 dy^2 + \dots + p_k dy^k \\ \text{subject to } \begin{array}{lllll} Ax + By^1 & & & & = b^1 \\ Ax & + By^2 & & & = b^2 \\ Ax & & + By^3 & & = b^3 \\ \vdots & & & \ddots & \vdots \\ Ax & & & + By^k & = b^k \end{array} \\ \quad x \geq 0, y^1 \geq 0, \dots, y^k \geq 0 \end{array}$$

Dual of the 2-stage stochastic LP problem:

$$\begin{array}{l} \text{Maximize } b^1u^1 + b^2u^2 + \dots + b^ku^k \\ \text{subject to } A^Tu^1 + A^Tu^2 + \dots + A^Tu^k \leq c \\ \quad B^Tu^1 \qquad\qquad\qquad \leq p_1 d \\ \qquad\quad B^Tu^2 \qquad\qquad \leq p_2 d \\ \qquad\qquad\quad \ddots \\ \qquad\qquad\qquad B^Tu^k \leq p_k d \\ \textit{all variables unrestricted in sign} \end{array}$$

This problem has a structure for which Dantzig-Wolfe decomposition is appropriate!

Dual of the 2-stage stochastic LP problem

$$\begin{array}{ll} \text{Maximize } b^1u^1 + b^2u^2 + \dots + b^ku^k & \\ \text{subject to } A^Tu^1 + A^Tu^2 + \dots + A^Tu^k \leq c & \leftarrow \text{linking constraints} \\ \left.\begin{array}{l} \quad B^Tu^1 \qquad\qquad\qquad \leq p_1 d \\ \qquad\quad B^Tu^2 \qquad\qquad \leq p_2 d \\ \qquad\qquad\quad \ddots \\ \qquad\qquad\qquad B^Tu^k \leq p_k d \end{array}\right\} & \text{subproblem constraints} \\ \textit{all variables unrestricted in sign} & \end{array}$$

Subproblem for Block # i

$$\begin{array}{ll} \text{Maximize} & (b^i - \omega A^T)u^i - \alpha_i \\ \text{subject to} & B^Tu^i \leq p_i d \end{array}$$

where ω is the simplex multiplier vector for the linking constraints,

and α_i is the simplex multiplier vector for convexity constraint # i

These subproblems all have the same matrix of constraint coefficients, and the constraint right-hand-side vectors are all scalar multiples of the same vector d.

Solution **Water Allocation Problem**

Define second-stage (recourse) variables

Y_i = amount of shortfall in water delivered to user i

maximize benefits minus expected penalties for shortfall

$$\text{Max } 100X_1 + 50X_2 + 30X_3 - E_Q\left\{\begin{array}{ll} \min & 250Y_1 + 75Y_2 + 60Y_3 \\ \text{s.t.} & Y_1 + Y_2 + Y_3 \geq X_1 + X_2 + X_3 - Q \\ & 0 \leq Y_1 \leq X_1,\ 0 \leq Y_2 \leq X_2,\ 0 \leq Y_3 \leq X_3 \end{array}\right\}$$

Define a separate recourse variable for each possible outcome:

Y_i^j = amount of shortfall in water delivered to user i if $Q = q_j$

In our "deterministic" LP formulation of the problem, then, we must simultaneously select the recourse (i.e., the user who will be denied the promised water) for each of the possible streamflows!

EQUIVALENT DETERMINISTIC LP

$$\text{Max } 100X_1 + 50X_2 + 30X_3 - 0.2(250Y_1^1 + 75Y_2^1 + 60Y_3^1) - 0.6(250Y_1^2 + 75Y_2^2 + 60Y_3^2) - 0.2(250Y_1^3 + 75Y_2^3 + 60Y_3^3)$$

subject to

$$\begin{cases} X_1 + X_2 + X_3 - Y_1^1 - Y_2^1 - Y_3^1 \le 4 \\ X_1 + X_2 + X_3 - Y_1^2 - Y_2^2 - Y_3^2 \le 10 \\ X_1 + X_2 + X_3 - Y_1^3 - Y_2^3 - Y_3^3 \le 17 \\ \left.\begin{array}{l} 0 \le Y_1^k \le X_1 \le 2 \\ 0 \le Y_2^k \le X_2 \le 3 \\ 0 \le Y_3^k \le X_3 \le 5 \end{array}\right\} \forall k=1,2,3 \end{cases}$$

Optimal Solution

Use i	Allocation X_i	Q=4 Shortfall in Delivery Y_i^1	10 Y_i^2	17 Y_i^3
1 Municipal	2	0	0	0
2 Industrial	3	1	0	0
3 Agricultural	5	5	0	0

Objective value = 100(2)+50(3)+30(5)−0.2[75(1)+60(5)]
= 500 − 0.2(375) = 425

Solution

	probability
0 : neither bid is successful	$(1-0.5)\times(1-0.40)= 0.30$
1 : bid #1 is successful, bid #2 is not	$0.5\times(1-0.40)= 0.30$
2 : bid #2 is successful, bid #1 is not	$(1-0.5)\times 0.60 = 0.30$
3 : both bids #1 and #2 are successful	$0.5\times 0.40 = 0.10$

Par, Inc.

Possible Outcomes ("scenarios")

Stage 1 Variables

X_1 = # standard bags in the next quarter's prod'n plan

X_2 = # deluxe bags in the next quarter's prod'n plan

Stage 2 Variables

For outcome # i (i=0,1,2,3)

Y^i = # standard bags added to next quarter's prod'n plan

T_{CD}^i = hours overtime in cut&dye

T_S^i = hours overtime in sewing

T_F^i = hours overtime in finishing

T_{IP}^i = hours overtime in inspect& pack

Scenario #0: neither bid is successful

Second-stage problem (X is fixed)

$$\text{Max } 8Y_1^0 - 5T_{CD}^0 - 6T_S^0 - 8T_F^0 - 4T_{IP}^0$$

subject to

$$\begin{aligned} \tfrac{7}{10}Y_1^0 - T_{CD}^0 &\le 630 - [\tfrac{7}{10}X_1 + X_2] \\ \tfrac{1}{2}Y_1^0 - T_S^0 &\le 600 - [\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2] \\ Y_1^0 - T_F^0 &\le 708 - [X_1 + \tfrac{2}{3}X_2] \\ \tfrac{1}{10}Y_1^0 - T_{IP}^0 &\le 135 - [\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2] \end{aligned}$$

$$Y_1^0 \ge 0,\ T_{CD}^0 \ge 0, T_{CD}^0 \ge 0, T_S^0 \ge 0, 100 \ge T_F^0 \ge 0, T_{IP}^0 \ge 0$$

Scenario #1: only bid #1 is successful

Second-stage problem (X is fixed)

$$\text{Max } 8Y_1^1 - 5T_{CD}^1 - 6T_S^1 - 8T_F^1 - 4T_{IP}^1$$

subject to

$$\begin{aligned} \tfrac{7}{10}Y_1^1 - T_{CD}^1 &\le 630 - 50 - [\tfrac{7}{10}X_1 + X_2] \\ \tfrac{1}{2}Y_1^1 - T_S^1 &\le 600 - 40 - [\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2] \\ Y_1^1 - T_F^1 &\le 708 - 80 - [X_1 + \tfrac{2}{3}X_2] \\ \tfrac{1}{10}Y_1^1 - T_{IP}^1 &\le 135 - 10 - [\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2] \end{aligned}$$

$$Y_1^1 \ge 0,\ T_{CD}^1 \ge 0, T_{CD}^1 \ge 0, T_S^1 \ge 0, 100 \ge T_F^1 \ge 0, T_{IP}^1 \ge 0$$

Scenario #2: only bid #2 is successful

Second-stage problem (X is fixed)

$$\text{Max } 8Y_1^2 - 5T_{CD}^2 - 6T_S^2 - 8T_F^2 - 4T_{IP}^2$$

subject to

$$\begin{aligned} \tfrac{7}{10}Y_1^2 - T_{CD}^2 &\le 630 - 30 - [\tfrac{7}{10}X_1 + X_2] \\ \tfrac{1}{2}Y_1^2 - T_S^2 &\le 600 - 50 - [\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2] \\ Y_1^2 - T_F^2 &\le 708 - 70 - [X_1 + \tfrac{2}{3}X_2] \\ \tfrac{1}{10}Y_1^2 - T_{IP}^2 &\le 135 - 15 - [\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2] \end{aligned}$$

$$Y_1^2 \ge 0,\ T_{CD}^2 \ge 0, T_{CD}^2 \ge 0, T_S^2 \ge 0, 100 \ge T_F^2 \ge 0, T_{IP}^2 \ge 0$$

Scenario #3: both bids are successful

Second-stage problem (X is fixed)

$$\text{Max } 8Y_1^3 - 5T_{CD}^3 - 6T_S^3 - 8T_F^3 - 4T_{IP}^3$$

$$\text{subject to } \tfrac{7}{10}Y_1^3 - T_{CD}^3 \le 630 - 50 - 30 - \left[\tfrac{7}{10}X_1 + X_2\right]$$

$$\tfrac{1}{2}Y_1^3 - T_S^3 \le 600 - 40 - 50 - \left[\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2\right]$$

$$Y_1^3 - T_F^3 \le 708 - 80 - 70 - \left[X_1 + \tfrac{2}{3}X_2\right]$$

$$\tfrac{1}{10}Y_1^3 - T_{IP}^3 \le 135 - 10 - 15 - \left[\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2\right]$$

$$Y_1^3 \ge 0,\ T_{CD}^3 \ge 0, T_{CD}^3 \ge 0, T_S^3 \ge 0, 100 \ge T_F^3 \ge 0, T_{IP}^3 \ge 0$$

Objective

$$\text{Max } 10X_1 + 9X_2 + 0.3\left(8Y_1^0 - 5T_{CD}^0 - 6T_S^0 - 8T_F^0 - 4T_{IP}^0\right)$$
$$+\ 0.3\left(8Y_1^1 - 5T_{CD}^1 - 6T_S^1 - 8T_F^1 - 4T_{IP}^1\right)$$
$$+\ 0.3\left(8Y_1^2 - 5T_{CD}^2 - 6T_S^2 - 8T_F^2 - 4T_{IP}^2\right)$$
$$+ 0.1\left(8Y_1^3 - 5T_{CD}^3 - 6T_S^3 - 8T_F^3 - 4T_{IP}^3\right)$$

Equivalent Deterministic Linear Programming Model

subject to

scenario #0
$$\left\{\begin{array}{r} \tfrac{7}{10}X_1 + X_2 + \tfrac{7}{10}Y_1^0 - T_{CD}^0 \le 630 \\ \tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 + \tfrac{1}{2}Y_1^0 - T_S^0 \le 600 \\ X_1 + \tfrac{2}{3}X_2 + Y_1^0 - T_F^0 \le 708 \\ \tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 + \tfrac{1}{10}Y_1^0 - T_{IP}^0 \le 135 \\ T_F^0 \le 100 \end{array}\right.$$

scenario #1
$$\left\{\begin{array}{r} \tfrac{7}{10}X_1 + X_2 + \tfrac{7}{10}Y_1^1 - T_{CD}^1 \le 580 \\ \tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 + \tfrac{1}{2}Y_1^1 - T_S^1 \le 560 \\ X_1 + \tfrac{2}{3}X_2 + Y_1^1 - T_F^1 \le 628 \\ \tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 + \tfrac{1}{10}Y_1^1 - T_{IP}^1 \le 125 \\ T_F^1 \le 100 \end{array}\right.$$

scenario #2
$$\left\{\begin{array}{r} \tfrac{7}{10}X_1 + X_2 + \tfrac{7}{10}Y_1^2 - T_{CD}^2 \le 600 \\ \tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 + \tfrac{1}{2}Y_1^2 - T_S^2 \le 550 \\ X_1 + \tfrac{2}{3}X_2 + Y_1^2 - T_F^2 \le 638 \\ \tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 + \tfrac{1}{10}Y_1^2 - T_{IP}^2 \le 120 \\ T_F^2 \le 100 \end{array}\right.$$

scenario #3
$$\left\{\begin{array}{r} \tfrac{7}{10}X_1 + X_2 + \tfrac{7}{10}Y_1^3 - T_{CD}^3 \le 550 \\ \tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 + \tfrac{1}{2}Y_1^3 - T_S^3 \le 510 \\ X_1 + \tfrac{2}{3}X_2 + Y_1^3 - T_F^3 \le 558 \\ \tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 + \tfrac{1}{10}Y_1^3 - T_{IP}^3 \le 110 \\ T_F^2 \le 100 \end{array}\right.$$

$$X_1 \ge 0,\ X_2 \ge 0,\ Y_1^i \ge 0,\ T_{CD}^i \ge 0, \quad T_{CD}^i \ge 0, T_S^i \ge 0, T_F^i \ge 0, T_{IP}^i \ge 0$$

i=0,1,2,3