SIGNOMIAL GEOMETRIC PROGRAMMING

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- Signomial Functions
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- Dual of Signomial GP
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Signomial Function

differs from a POSYNOMIAL FUNCTION in that the coefficients need not be positive.

$$g_k(x) = \sum_{i\in[k]} \sigma_i c_i \prod_{j=1}^{m} x_j^{a_{ij}}$$

where c_i = absolute value of coefficient
σ_i = sign of coefficient (+1 or −1)

EXAMPLE

$$g_0(x) = 2x_1^2x_2^{-1} - 5x_1x_2$$

For this signomial:

$\sigma_1 = +1 \quad \sigma_2 = -1$

$c_1 = 2 \quad c_2 = 5$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Primal Signomial GP Problem

Minimize $g_0(x_1, x_2, \cdots x_m)$

subject to $g_k(x_1, x_2, \cdots x_m) \le \zeta_k$, $k=1, 2, \cdots p$

$x_j > 0$, $j=1,2,\cdots m$

where $g_k(x) = \sum_{i\in[k]} \sigma_i c_i \prod_{j=1}^{m} x_j^{a_{ij}}$

signs
$\sigma_i = \pm 1 \ \forall i$
$\zeta_k = \pm 1 \ \forall k$

EXAMPLE

Minimize $-2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$

subject to $-x_4^{1/2} + x_2^{1/3}x_3 \le -1$

$x_1, x_2, x_3, x_4 > 0$

$\sigma = [-1, +1, +1, -1, +1]$

$c = [2, \ 1, \ 5, \ 1, \ 1]$

$\zeta_0 = \ ?$ *(sign of objective at optimum)*

$\zeta_1 = -1$

$$A = \begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & -1 & -1 & 0 \\ 1/2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1 & 0 \end{bmatrix}$$

"pseudo" Dual Signomial GP Problem

Maximize $\zeta_0 \left[\prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\sigma_i \delta_i} \prod_{k=1}^{p} \lambda_k^{\zeta_k \lambda_k} \right]^{\zeta_0}$

subject to

$\sum_{i\in[k]} \sigma_i \delta_i = \zeta_k \lambda_k$, $k=0, 1, \cdots p$

$\sum_{i=1}^{n} \sigma_i a_{ij} \delta_i = 0$, $j=1,2,\cdots m$

$\delta_i \ge 0$, $i=1, 2, \cdots n$

$\lambda_0 = 1$

$\lambda_k \ge 0$, $k=1,2,\cdots p$

For every locally minimum $\hat{x}$ of the primal signomial GP problem, there exists a dual feasible solution $(\hat{\delta}, \hat{\lambda})$ and sign ζ_0 such that

$$g_0(\hat{x}) = v(\hat{\delta}, \hat{\lambda})$$

Furthermore, $\hat{x}$ and $(\hat{\delta}, \hat{\lambda})$ are related by

$$c_i \prod_j \hat{x}_j^{a_{ij}} = \zeta_0 \hat{\delta}_i g_0(\hat{x}) \quad \forall\, i \in [0]$$

$$c_i \prod_j \hat{x}_j^{a_{ij}} = \frac{\hat{\delta}_i}{\hat{\lambda}_k} \quad \forall\, i \in [k],\ k \ge 1$$

A *Weak Duality* theorem would say that

$$g_0(x) \geq v(\delta, \lambda)$$

for any primal-feasible x and dual-feasible δ, λ

This is *NOT* true of the pseudo-dual signomial GP problem!

EXAMPLE

Minimize $-2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$

subject to $-x_4^{1/2} + x_2^{1/3}x_3 \leq -1$

$x_1, x_2, x_3, x_4 > 0$

$$\text{Max} \quad \zeta_0\left[\left(\frac{2}{\delta_1}\right)^{-\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{5}{\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{-\delta_4}\left(\frac{1}{\delta_5}\right)^{\delta_5}\lambda_1^{-\lambda_1}\right]^{\zeta_0}$$

pseudo-dual objective

$$\delta_i \geq 0,\ i=1,\cdots 5;\ \lambda_1 \geq 0$$

EXAMPLE

Minimize $-2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$

subject to $-x_4^{1/2} + x_2^{1/3}x_3 \leq -1$

$x_1, x_2, x_3, x_4 > 0$

$$-\delta_1 + \delta_2 + \delta_3 = \zeta_0$$
$$-\delta_4 + \delta_5 = -\lambda_1$$

EXAMPLE

Minimize $-2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$

subject to $-x_4^{1/2} + x_2^{1/3}x_3 \leq -1$

$x_1, x_2, x_3, x_4 > 0$

$$-\delta_1 + \tfrac{1}{2}\delta_3 = 0$$
$$-\delta_1 + \delta_2 + \tfrac{1}{3}\delta_5 = 0$$
$$-4\delta_1 - \delta_2 + \delta_5 = 0$$
$$\delta_1 + \delta_3 - \tfrac{1}{2}\delta_4 = 0$$

Orthogonality Constraints

$$\text{Maximize} \quad \zeta_0\left[\left(\frac{2}{\delta_1}\right)^{-\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{5}{\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{-\delta_4}\left(\frac{1}{\delta_5}\right)^{\delta_5}\lambda_1^{-\lambda_1}\right]^{\zeta_0}$$

subject to

$$-\delta_1 + \delta_2 + \delta_3 = \zeta_0$$
$$-\delta_4 + \delta_5 = -\lambda_1$$
$$-\delta_1 + \tfrac{1}{2}\delta_3 = 0$$
$$-\delta_1 + \delta_2 + \tfrac{1}{3}\delta_5 = 0$$
$$-4\delta_1 - \delta_2 + \delta_5 = 0$$
$$\delta_1 + \delta_3 - \tfrac{1}{2}\delta_4 = 0$$
$$\delta_i \geq 0,\ i=1,\cdots 5;\ \lambda_1 \geq 0$$

Pseudo-Dual Signomial GP Problem

\# terms = 5;
\# primal variables = 4 $\Rightarrow$ degree of difficulty = zero

Let's guess that $\zeta_0 = +1$, i.e., that the optimal primal objective is positive...
The unique solution of the dual constraints is

$$\delta_1^* = 2/3,\ \delta_2^* = 1/3,\ \delta_3^* = 4/3,\ \delta_4^* = 4,\ \delta_5^* = 3,\ \lambda_1^* = 1$$

Since $\delta_i \geq 0$ & $\lambda_1 \geq 0$, the guess $\zeta_0 = +1$ was correct.

$$v(\delta^*,\lambda^*) = \left(\frac{2}{2/3}\right)^{-2/3}\left(\frac{1}{1/3}\right)^{1/3}\left(\frac{5}{4/3}\right)^{4/3}\left(\frac{1}{4}\right)^{-4}\left(\frac{1}{3}\right)^{3}1^{-1} = 38.322$$

Computing the Primal Optimum

$$2x_1x_2x_3^4x_4^{-1} = \delta_1^*g_0(x^*) = 2/3 \times (38.322) \Rightarrow x_1x_2x_3^4x_4^{-1} = 12.77$$
$$x_2^{-1}x_3^{-1} = \delta_2^*g_0(x^*) = 1/3 \times (38.322) \Rightarrow x_2^{-1}x_3^{-1} = 12.77$$
$$5x_1^{1/2}x_4 = \delta_3^*g_0(x^*) = 4/3 \times (38.322) \Rightarrow x_1^{1/2}x_4 = 10.22$$
$$x_4^{1/2} = \delta_4^*/\lambda_1^* = 4 \Rightarrow x_4^{1/2} = 4$$
$$x_2^{1/3}x_3 = \delta_5^*/\lambda_1^* = 3 \Rightarrow x_2^{1/3}x_3 = 3$$

5 nonlinear equations with 4 unknowns

Computing the Primal Optimum

Taking logarithms of each side yields equations which are linear in the logarithms of the primal variables:

$$\begin{cases} \ln x_1 + \ln x_2 + 4\ln x_3 - \ln x_4 = \ln 12.77 \\ -\ln x_2 - \ln x_3 = \ln 12.77 \\ \tfrac{1}{2}\ln x_1 + \ln x_4 = \ln 10.22 \\ \tfrac{1}{2}\ln x_4 = \ln 4 \\ \tfrac{1}{3}\ln x_2 + \ln x_3 = \ln 3 \end{cases}$$

Computing the Primal Optimum

$$\begin{cases} x_1^* = 0.4079 \\ x_2^* = 0.004216 \\ x_3^* = 18.57 \\ x_4^* = 16 \end{cases}$$

Example

Maximize $5x_1^2 - x_2^2 x_3^4$
subject to

$$\frac{-5x_1^2}{x_2^2} + \frac{3\,x_3}{x_2} \geq 2$$

$$x_1 > 0,\ x_2 > 0,\ x_3 > 0$$