Set Covering Problem

Lagrangian Relaxation Algorithm

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Lagrangian Relaxation

Define a multiplier $\lambda_i \geq 0$ for each point i

The Lagrangian relaxation is

$$\text{Minimize } \sum_{j=1}^{N} C_j X_j + \sum_{i=1}^{M} \lambda_i \left(1 - \sum_{j=1}^{N} a_{ij} X_j\right)$$

$$\text{subject to } \quad X_j \,\varepsilon\, \{0,1\} \quad \text{for each } j=1, 2, \ldots N$$

$$\Phi(\lambda) = \text{Minimum } \sum_{j=1}^{N} C_j X_j + \sum_{i=1}^{M} \lambda_i \left(1 - \sum_{j=1}^{N} a_{ij} X_j\right)$$

$$\text{subject to } \quad X_j \,\varepsilon\, \{0,1\} \quad \text{for each } j=1, 2, \ldots N$$

$$\Phi(\lambda) = \underset{X_j \in \{0,1\}}{\text{Minimum}} \sum_{j=1}^{N} \left(C_j - \sum_{i=1}^{M} \lambda_i a_{ij}\right) X_j + \sum_{i=1}^{M} \lambda_i$$

Interpretation: We no longer require that each point be covered. Rather, we offer an incentive λ_i for each point i which is covered.

The Lagrangian relaxation is trivial to solve:

$$\Phi(\lambda) = \underset{X_j \in \{0,1\}}{\text{Minimum}} \sum_{j=1}^{N} \left(C_j - \sum_{i=1}^{M} \lambda_i a_{ij}\right) X_j + \sum_{i=1}^{M} \lambda_i$$

$$= \sum_{j=1}^{N} \text{Minimum} \left\{0, C_j - \sum_{i=1}^{M} \lambda_i a_{ij}\right\} + \sum_{i=1}^{M} \lambda_i$$

That is, we select set #j if its "reduced cost", $\bar{C}_j \equiv C_j - \sum_{i=1}^{M} \lambda_i a_{ij}$, is ≤ 0.

Suppose that X^* is the optimal solution of the original set-covering problem. Then for any $\lambda \geq 0$,

$$\Phi(\lambda) = \underset{X_j \in \{0,1\}}{\text{Minimum}} \sum_{j=1}^{N} C_j X_j + \sum_{i=1}^{M} \lambda_i \left(1 - \sum_{j=1}^{N} a_{ij} X_j\right)$$

$$\leq \sum_{j=1}^{N} C_j X_j^* + \sum_{i=1}^{M} \lambda_i \underbrace{\left(1 - \sum_{j=1}^{N} a_{ij} X_j^*\right)}_{\leq 0}$$

$$\leq \sum_{j=1}^{N} C_j X_j^* = Z^*$$

That is, the solution of the Lagrangian relaxation provides a lower bound on Z!*

Lagrangian Dual Problem

For each $\lambda \geq 0$, $\Phi(\lambda)$ is a lower bound on the optimum of the SCP.

The Lagrangian Dual problem is to select λ so as to obtain the *greatest* lower bound:

$$\underset{\lambda \geq 0}{\text{Maximize}} \; \Phi(\lambda)$$

What is the nature of the dual objective function $\Phi(\lambda)$?

In solving the Lagrangian relaxation, we must choose from among 2^N different collections of sets, each specified by a vector $\hat{X}^k$, $k=1, 2, \ldots 2^N$

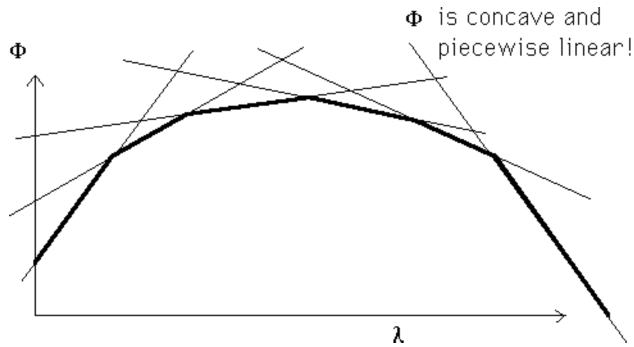
$$\Phi(\lambda) = \underset{X_j \in \{0,1\}}{\text{Minimum}} \sum_{j=1}^{N} C_j X_j + \sum_{i=1}^{M} \lambda_i \left(1 - \sum_{j=1}^{N} a_{ij} X_j\right)$$

$$= \underset{k=1,2,\ldots 2^N}{\text{minimum}} \left\{ \sum_{j=1}^{N} C_j \hat{X}_j^k + \sum_{i=1}^{M} \lambda_i \left(1 - \sum_{j=1}^{N} a_{ij} \hat{X}_j^k\right) \right\}$$

$$\Phi(\lambda) = \underset{k=1,2,\ldots 2^N}{\text{minimum}} \left\{ \sum_{j=1}^{N} C_j \hat{X}_j^k + \sum_{i=1}^{M} \lambda_i \left(1 - \sum_{j=1}^{N} a_{ij} \hat{X}_j^k\right) \right\}$$

$$= \underset{k=1,2,\ldots 2^N}{\text{minimum}} \left\{ \sum_{i=1}^{M} \hat{\alpha}_i^k \lambda_i + \hat{\beta}^k \right\}$$

$\Phi(\lambda)$ is the minimum (the "lower envelope") of a set of 2^N linear functions!



Subgradient Optimization

One method of searching for the optimal dual variables is to step in the direction of a subgradient of the dual objective function, $\Phi(\lambda)$

At iteration t, let λ^t be the current value of the dual variables, and let $\hat{X}^t$ be the solution of the Lagrangian relaxation. That is,

$$\Phi(\lambda^t) = \sum_{j=1}^{N} C_j \hat{X}_j^t + \sum_{i=1}^{M} \lambda_i^t \left(1 - \sum_{j=1}^{N} a_{ij} \hat{X}_j^t\right)$$

Then Δ is a subgradient of Φ at λ^t, with elements

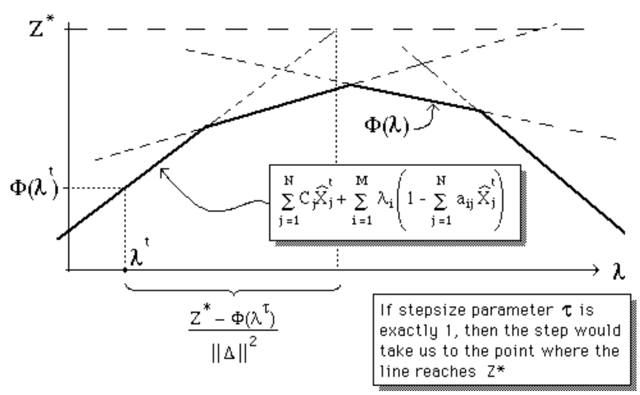
$$\delta_i = \left(1 - \sum_{j=1}^{N} a_{ij} \hat{X}_j^t\right)$$

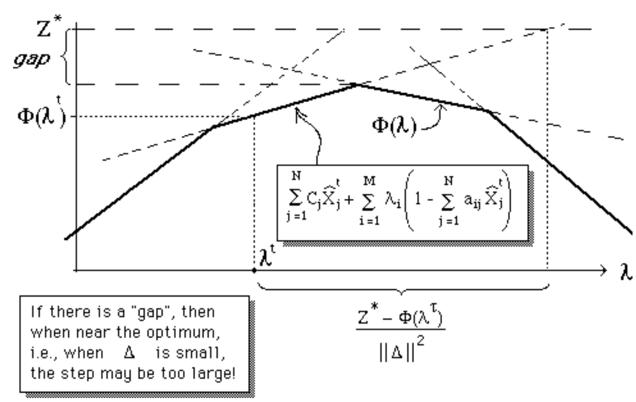
Let the step taken be $\lambda^{t+1} = \lambda^t + \alpha\Delta$

Rather than selecting the stepsize α by a one-dimensional search method
(e.g., golden-section or Fibonacci search, which requires many evaluations of the function $\Phi(\lambda + \alpha\Delta)$),
"subgradient optimization" selects the stepsize

$$\alpha = \tau \frac{Z^* - \Phi(\lambda^t)}{\|\Delta\|^2}$$

where Z^* is an upper bound, and τ is in the interval (0,2]





Dual Ascent

Subgradient optimization is guaranteed to converge to the optimal dual value, but not monotonically.

In "dual ascent", the dual objective increases monotonically at each iteration (i.e., never decreases). However, it may not converge to the optimal dual value.

Dual Ascent

Choose a multiplier whose subgradient element is nonzero, i.e., λ_i such that $0 \neq \delta_i = \left(1 - \sum_{j=1}^{N} a_{ij} \hat{X}_j^t\right)$

(This means that point # i is not covered, if $\delta_i = 1$*, or else covered by more than one set if* $\delta_i < 0$*)*

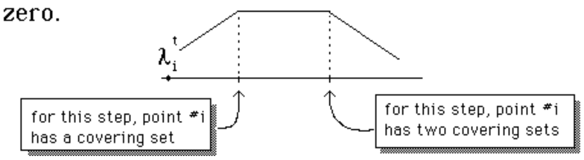
If a multiplier is adjusted upward or downward by an amount which leaves the signs of the reduced costs unchanged, then the $\hat{X}$ solving the Lagrangian relaxation is unchanged, although $\Phi(\lambda)$ changes.

$$\Phi(\lambda) = \sum_{j=1}^{N} C_j \hat{X}_j^t + \sum_{i=1}^{M} \lambda_i \left(1 - \sum_{j=1}^{N} a_{ij} \hat{X}_j^t\right)$$

$$\Phi(\lambda) = \sum_{j=1}^{N} C_j \hat{X}_j^t + \sum_{i=1}^{M} \lambda_i \delta_i$$

Consider first the case $\delta_i = +1$,
i.e., point #i is uncovered.

As we increase λ_i, the dual objective function increases with slope +1, until the reduced cost of a set which covers i becomes zero or negative. At that value of λ_i, the slope of $\Phi(\lambda)$ becomes zero.



In the case $\delta_i < 0$, point i has multiple covering sets:

As we decrease λ_i, the dual objective function increases, and the reduced cost of each set covering point i increases. Sets drop out of the solution when their reduced cost reaches zero.