Examples: Unconstrained Optimization

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

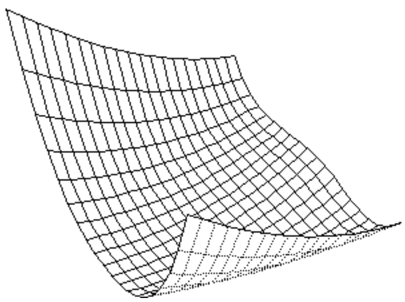
author

$$f(x_1,x_2) = (x_2 - x_1^2)^2 + (1 - x_1)^2$$

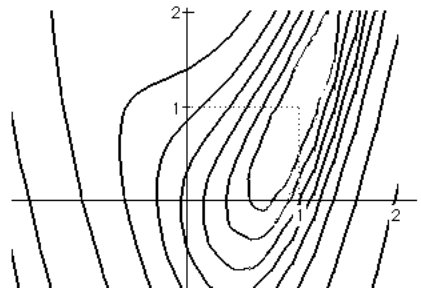
```
    ∇Z←F X
[1]   ⍝
[2]   ⍝        TEST FUNCTION FOR UNCONSTRAINED MINIMIZATION
[3]   ⍝
[4]   f_evaluations←f_evaluations+1
[5]    Z←((X[2]-X[1]*2)*2)+(1-X[1])*2
    ∇
```

$$\nabla f(x_1,x_2) = \begin{bmatrix} 4x_1^3 - 4x_1x_2 + 2x_1 - 2 \\ 2\,(x_2 - x_1^2) \end{bmatrix}$$

```
    ∇G←GRADIENT X
[1]   ⍝
[2]   ⍝      GRADIENT OF TEST FUNCTION FOR
[3]   ⍝        UNCONSTRAINED MINIMIZATION
[4]   ⍝
[5]   gradient_evaluations←gradient_evaluations+1
[6]    G←(4×X[1]*3)+(¯4×X[1]×X[2])+(2×X[1])-2
[7]    G←G,2×(X[2]-X[1]*2)
    ∇
```

Contours of $f(x_1,x_2) = (x_2 - x_1^2)^2 + (1 - x_1)^2$



$f(x_1,x_2) \geq 0$,
& $f(1,1) = 0$
$\Rightarrow x^* = (1,1)$

Tolerances for Search

Stopping Criteria:

One-dimensional search (max |derivative|):	0.001
One-dimensional search (|Δx|):	0.01
Maximum Abs. Value of partial derivatives:	0.001
Improvement in objective function:	0.00001
Length of step:	0.001
Simplex: stopping criterion:	0.0001
alpha	1.1
beta	0.5
gamma	1.5

Maximum number of iterations

25 for 1-dimensional search
50 for search algorithm

Numerical Example

- ☞ Newton's Method
- ☞ Newton's Method with Linesearch
- ☞ Steepest Descent Method
- ☞ Fletcher-Reeves Conjugate Gradient Method
- ☞ Davidon-Fletcher-Powell (DFP) Method
- ☞ Powell's Method

Newton's Algorithm

Minimize $f(x_1,x_2) = (x_2 - x_1^2)^2 + (1 - x_1)^2$

The optimum is at $x^* = (1,1)$

We will use $x^\circ = (-2,+2)$

Newton's Algorithm

Iteration 1

x = ¯2 2
F(x) = 13
∇F(x) = ¯22 ¯4

Hessian Matrix = $\begin{bmatrix} 12 & 8 \\ 8 & 2 \end{bmatrix}$

Step: Δx= 0.6 ¯0.4,
with magnitude 0.7211102551

$\begin{bmatrix} 12 & 8 \\ 8 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -22 \\ -4 \end{bmatrix}$

Iteration 2

```
x = ¯1.4 1.6
F(x) = 5.8896
Improvement: 7.1104
∇F(x) = ¯6.816 ¯0.72

Hessian Matrix = ⎡19.12   5.6⎤
                 ⎣ 5.6    2  ⎦

Step: Δx= 1.395348837 ¯3.546976744,
  with magnitude 3.811566922
```

$x^0 + \Delta x = (-2, 2) + (0.6, -0.4)$

Iteration 3

```
x = ¯0.004651162791 ¯1.946976744
F(x) = 4.800126641
Improvement: 1.089473359
∇F(x) = ¯2.045525551 ¯3.893996755

Hessian Matrix = ⎡9.788166577     0.01860465116⎤
                 ⎣0.01860465116   2            ⎦

Step: Δx= 0.2052823516 1.945088774,
  with magnitude 1.955891404
```


Iteration 4

```
x = 0.2006311889 ¯0.001887969956
F(x) = 0.640766347
Improvement: 4.159360294
∇F(x) = ¯1.564918552 ¯0.08428168779

Hessian Matrix = ⎡ 2.490586367   ¯0.8025247554⎤
                 ⎣¯0.8025247554   2           ⎦

Step: Δx= 0.7372335253 0.3379649212,
  with magnitude 0.8110077428
```


Iteration 5

```
x = 0.9378647142 0.3360769512
F(x) = 0.2992674694
Improvement: 0.3414988776
∇F(x) = 1.914697102 ¯1.087026542

Hessian Matrix = ⎡11.21077486    ¯3.751458857⎤
                 ⎣¯3.751458857    2          ⎦

Step: Δx= 0.02977215889 0.5993577855,
  with magnitude 0.6000967726
```


Iteration 6

```
x = 0.9676368731 0.9354347367
F(x) = 0.001048157656
Improvement: 0.2982193117
∇F(x) = ¯0.06129547236 ¯0.00177276289

Hessian Matrix = ⎡ 9.494114471  ¯3.870547492⎤
                 ⎣¯3.870547492   2          ⎦

Step: Δx= 0.0323058563 0.06340705698,
  with magnitude 0.07116265331
```


Iteration 7

```
x = 0.9999427294 0.9988417937
F(x) = 1.092523551E¯6
Improvement: 0.001047065133
∇F(x) = 0.004059893072 ¯0.002087336702

Hessian Matrix = ⎡10.00325837    ¯3.999770918⎤
                 ⎣¯3.999770918    2          ⎦

Step: Δx= 0.00005715132911 0.001157964463,
  with magnitude 0.001159373957
```


Iteration 8

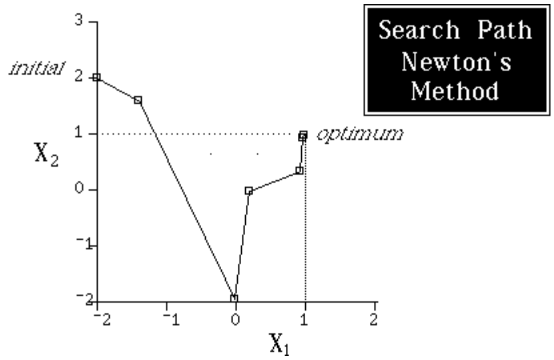
```
x = 0.9999998807 0.9999997581
F(x) = 1.424174293E¯14
Improvement: 1.092523537E¯6
∇F(x) = ¯2.25523038E¯7 ¯6.532548724E¯9

Hessian Matrix = ⎡ 9.999998104 ¯3.999999523⎤
                 ⎣¯3.999999523  2          ⎦
```

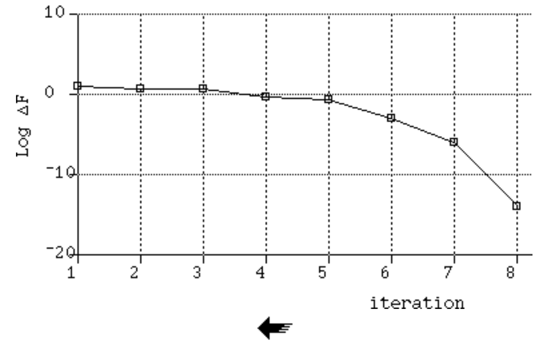
Convergence criterion satisfied

```
Improvement in objective
Gradient
*** CONVERGED ***
```



```
Solution found is 0.9999998807 0.9999997581
where F is 1.424174293E¯14
and ∇F is ¯2.25523038E¯7 ¯6.532548724E¯9

# iterations = 8
# function evaluations= 9
# gradient evaluations= 9
# hessian evaluations= 8
Elapsed CPU time: 23.95 seconds
```





Newton's Method with Linesearch

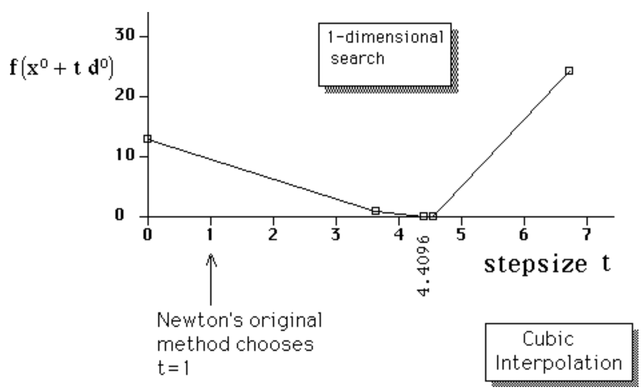
Newton's Method

(incorporating 1-dimensional search)

Iteration 1

```
x = ¯2 2
F(x) = 13
∇F(x) = ¯22 ¯4

Hessian Matrix = [ 42  8 ]
                 [  8  2 ]

Search direction is 0.6 ¯0.4
Optimal stepsize is 4.40964343
Step: Δx= 2.645786058 ¯1.763857372,
  with magnitude 3.179839099
```

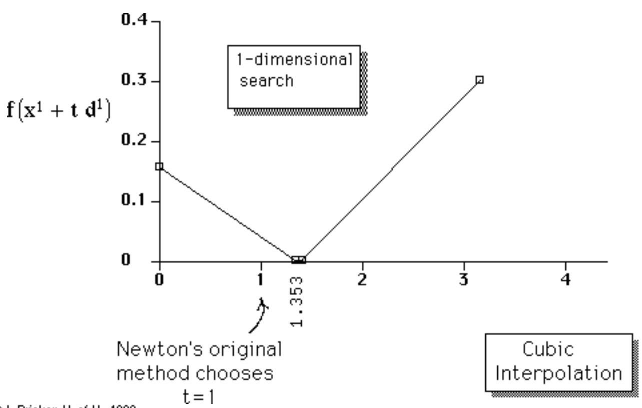
Much greater than that chosen by Newton's method (t=1)



Iteration 2

```
x = 0.6457860579 0.236142628
F(x) = 0.158191243
Improvement: 12.84180876
∇F(x) = ¯0.2411448302 ¯0.3617940092

Hessian Matrix = [  6.059905079  ¯2.583144232 ]
                 [ ¯2.583144232   2           ]

Search direction is 0.2601083128 0.5168456485

Optimal stepsize is 1.353002541

Step: Δx= 0.3519272081 0.6992934756,
  with magnitude 0.7828563884
```

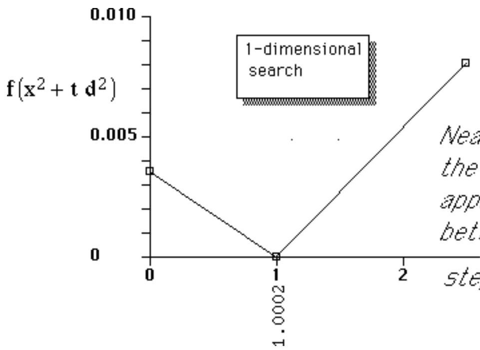
As we get nearer to the optimum, the optimal step size gets nearer to 1.0



Iteration 3

```
x = 0.997713266 0.9354361037
F(x) = 0.003604708074
Improvement: 0.1545865349
∇F(x) = 0.2348603857 ¯0.119991315

Hessian Matrix = [ 10.20343672  ¯3.990853064 ]
                 [ ¯3.990853064  2           ]

Search direction is 0.002041742593 0.06406980486
Optimal stepsize is 1.000208261
Step: Δx= 0.002042167809 0.06408314813,
  with magnitude 0.06411567923
```

$t \approx 1$



Near the optimum, the quadratic approximation is better, and the stepsize ≈ 1

Iteration 4

```
x = 0.9997554338 0.9995192518
F(x) = 5.988190232E¯8
Improvement: 0.003604648192
∇F(x) = ¯0.0005224214986 0.00001664865847
```

Hessian Matrix = $\begin{bmatrix} 9.996054123 & -3.999021735 \\ -3.999021735 & 2 \end{bmatrix}$

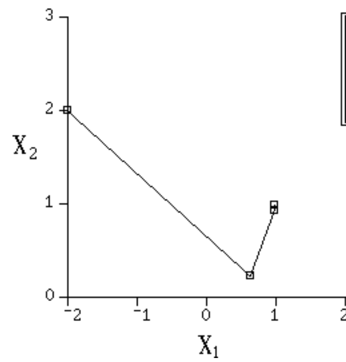
Convergence criterion satisfied:
Gradient

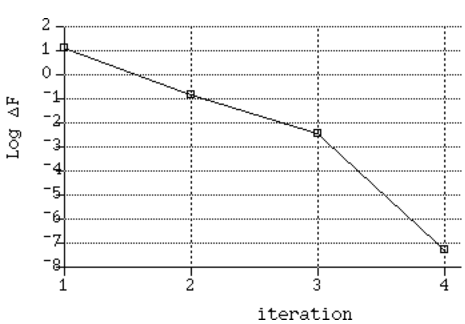
Convergence criterion satisfied:
Gradient

*** CONVERGED ***

```
Solution found is 0.9997554338 0.9995192518
where F is 5.988190232E¯8
and ∇F is ¯0.0005224214986 0.00001664865847

# iterations = 4
# function evaluations= 37
# gradient evaluations= 21
# hessian evaluations= 4
Elapsed CPU time: 26.65 seconds
```


Search Path Newton's Method with Linesearch





Steepest Descent

Steepest Descent Algorithm

Iteration 1

```
x = ¯2 2
F(x) = 13
∇F(x) = ¯22 ¯4

Search direction is -∇F = 22 4
Optimal stepsize is 0.1621737068
Step is Δx= 3.567821549 0.6486948271,
  with magnitude 3.626314325
```


Iteration 2

```
x = 1.567821549 2.648694827
F(x) = 0.3587612676
Improvement: 12.64123873
∇F(x) = ¯0.05985480199 0.3812608333

Search direction is -∇F = 0.05985480199 ¯0.3812608333
Optimal stepsize is 0.2275651301
Step is Δx= 0.0136208658 ¯0.08676167111,
  with magnitude 0.08782434491
```

$$\begin{bmatrix} 22 \\ 4 \end{bmatrix}^T \begin{bmatrix} -0.05985480199 \\ 0.3812608333 \end{bmatrix} = -0.208238$$

Iteration 3

```
x = 1.581442415 2.561933156
F(x) = 0.3417929942
Improvement: 0.01696827346
∇F(x) = 0.7771834003 0.1219460874
Search direction is -∇F = ¯0.7771834003 ¯0.1219460874
Optimal stepsize is 0.05369486991
Step is Δx= ¯0.04173076157 ¯0.006547879298,
  with magnitude 0.04224134449
```

$$\begin{bmatrix} -0.05985480199 \\ 0.3812608333 \end{bmatrix}^T \begin{bmatrix} 0.7771834003 \\ 0.1219460874 \end{bmatrix} = 0.000025$$

Iteration 4

```
x = 1.539711654 2.555385277
F(x) = 0.325392897
Improvement: 0.0164000972
∇F(x) = ¯0.05795122546 0.3693466013
Search direction is -∇F = 0.05795122546 ¯0.3693466013
Optimal stepsize is 0.2303221653
Step is Δx= 0.01334745173 ¯0.08506870896,
  with magnitude 0.08610946354
```

.... *etc.*

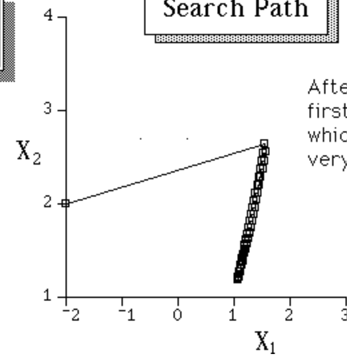
Steepest Descent

After fifty iterations, the current solution is still relatively far from optimal!

```
*** Warning: did not converge in 50 steps

Solution found is 1.080545748 1.178896815
where F is 0.006615707827
and ∇F is 0.1121743152 0.02263540436

# iterations = 51
# function evaluations= 320
# gradient evaluations= 263
Elapsed CPU time: 125.25 seconds
```


Steepest Descent

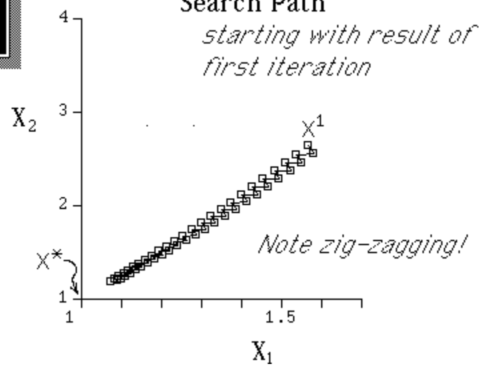
Search Path

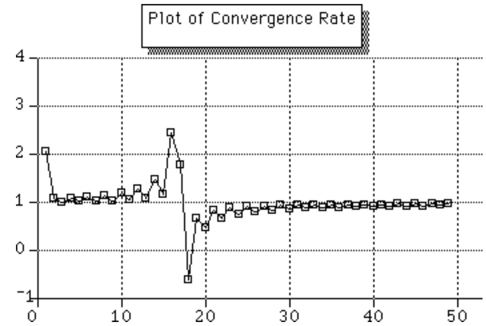


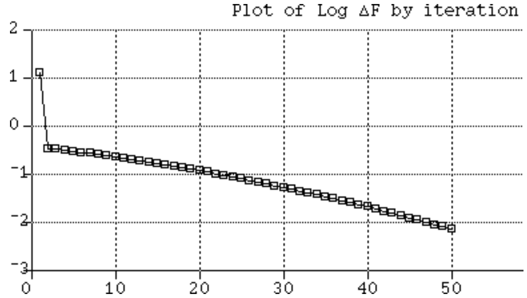
After a rather large first step, the steps which follow are very small!

Steepest Descent

Search Path







Fletcher-Reeves Algorithm

(Also known as "Conjugate Gradient" method)

The step direction is a combination of the steepest descent direction and the previous search direction:

$$d^k = -\nabla f(x^k) + \frac{\left\|\nabla f(x^k)\right\|^2}{\left\|\nabla f(x^{k-1})\right\|^2} d^{k-1}$$

Fletcher-Reeves
Conjugate-Gradient
Algorithm

Iteration 1

x = ¯2 2
F(x) = 13
∇F(x) = ¯22 ¯4 ← $\|\nabla f(x^0)\|^2 = 500$

New search direction is 22 4 ← *initially, d is steepest descent direction*

Optimal stepsize is 0.1621737068

Step: Δx= 3.567821549 0.6486948271,
with magnitude 3.626314325

Iteration 2

x = 1.567821549 2.648694827
F(x) = 0.3587612676
Improvement is 12.64123873
∇F(x) = ¯0.05985480199 0.3812608333 ← $\|\nabla f(x^1)\|^2 = 0.14894$

Multiplier for old search direction = 0.0002978848406 ← $\frac{0.14894}{500}$

New search direction is 0.06640826848 ¯0.3800692939
(= (-∇F) + 0.0002978848406 × (old search direction))

Optimal stepsize is 0.2124445147

Step: Δx= 0.01410807237 ¯0.08074363668,
with magnitude 0.08196689924

Iteration 3

x = 1.581929622 2.56795119
F(x) = 0.3429257691
Improvement is 0.01583549852
∇F(x) = 0.7497109385 0.130899725 — $\|\nabla f(x^1)\|^2 = 0.5792$

Multiplier for old search direction = 3.888759348 ← $\frac{0.5792}{0.14894}$

New search direction is ¯0.4914651637 ¯1.608897745
(= (-∇F) + 3.888759348 × (old search direction))

Optimal stepsize is 0.8659182831

Step: Δx= ¯0.4255686707 ¯1.393173973,
with magnitude 1.456723176

Iteration 4

x = 1.156360951 1.174777218
F(x) = 0.05082037349
Improvement is 0.2921053956

∇F(x) = 1.063863592 ¯0.3247868624

Multiplier for old search direction = 2.136204459

New search direction is ¯2.113733666 ¯3.112147673
(= (-∇F) + 2.136204459 × (old search direction))

Optimal stepsize is 0.0841505677

Step: Δx= ¯0.1778718879 ¯0.2618889935,
with magnitude 0.3165821432

Iteration 5

x = 0.978489063 0.9128882243
F(x) = 0.00244765655
Improvement is 0.04837271694

∇F(x) = 0.1313551401 ¯0.08910524431

Multiplier for old search direction = 0.02036213953

New search direction is ¯0.17439528 0.02573525915
(= (-∇F) + 0.02036213953 × (old search direction))

Optimal stepsize is 0.07623555685

Step: Δx= ¯0.01329512128 0.001961941812,
with magnitude 0.01343910211

Iteration 6

x = 0.9651939418 0.9148501662
F(x) = 0.001491996689
Improvement is 0.0009556598605

∇F(x) = ¯0.004947291822 ¯0.03349835813

Multiplier for old search direction = 0.04551160803

New search direction is ¯0.002989717803 0.03466961116
(= (-∇F) + 0.04551160803 × (old search direction))

Optimal stepsize is 0.3486481657

Step: Δx= ¯0.001042359628 0.01208749634,
with magnitude 0.01213235679

Iteration 7

x = 0.9641515821 0.9269376625
F(x) = 0.001292134801
Improvement is 0.0001998618881

∇F(x) = ¯0.06147447316 ¯0.005301221693

Multiplier for old search direction = 3.320392197

New search direction is 0.0515474375 0.120417928
(= (-∇F) + 3.320392197 × (old search direction))

Optimal stepsize is 0.6319461373

Step: Δx= 0.03257520401 0.07609764448,
with magnitude 0.08277678064

Iteration 8

x = 0.9967267862 1.003035307
F(x) = 0.0001023183669
Improvement is 0.001189816434

∇F(x) = ¯0.04470519867 0.01914204148

Multiplier for old search direction = 0.6211819624

New search direction is 0.07672553705 0.05565940336
(= (-∇F) + 0.6211819624 × (old search direction))

Optimal stepsize is 0.07691320942

Step: Δx= 0.005901207299 0.004280943346,
with magnitude 0.007290454274

Iteration 9

```
x = 1.002627993 1.00731625
F(x) = 0.00001112262483
Improvement is 0.00009119574211

∇F(x) = ¯0.002979026213 0.004106714139

Multiplier for old search direction = 0.01088371968

New search direction is 0.003814085451 ¯0.003500932795
  (= (-∇F) + 0.01088371968 × (old search direction) )

Optimal stepsize is 0.09269772415

Step: Δx= 0.000353557041 ¯0.0003245285025,
  with magnitude 0.0004799180453
```


Iteration 10

```
x = 1.00298155 1.006991722
F(x) = 9.929495032E¯6
Improvement is 1.193129795E¯6

∇F(x) = 0.001872014698 0.002039462382

Multiplier for old search direction = 0.2977441996
```

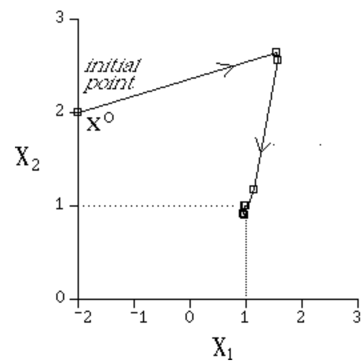
Convergence criterion satisfied:

Improvement in objective
Step size

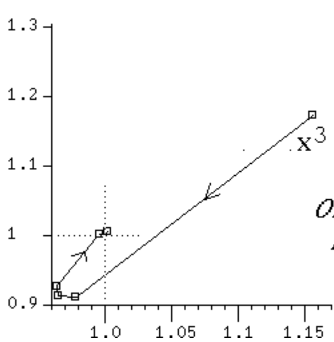
Convergence criterion satisfied:

Improvement in objective
Step size

```
*** CONVERGED ***

Solution found is 1.00298155 1.006991722

where F is 9.929495032E¯6
and ∇F is 0.001872014698 0.002039462382

# iterations = 10
# function evaluations= 80
# gradient evaluations= 51
Elapsed CPU time: 52.65 seconds
```

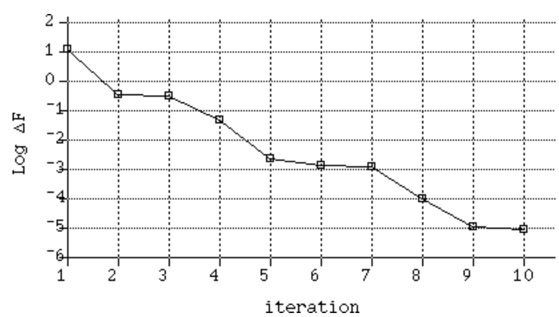



Search Path Fletcher-Reeves Conjugate Gradient Algorithm



Search Path Fletcher-Reeves Conjugate Gradient Algorithm

Omitting x^0, x^1, & x^2 from the plot



Davidon-Fletcher-Powell Algorithm

Iteration 1

```
x = ¯2 2
F(x) = 13
∇F(x) = ¯22 ¯4
```

Q (approx. of Hessian inverse) Actual Hessian inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0.1 & -0.4 \\ -0.4 & 2.1 \end{bmatrix}$$

```
Search direction is 22 4
Optimal stepsize is 0.1621737068
Step is Δx= 3.567821549 0.6486948271,
  with magnitude 3.626314325
Δgradient = 21.9401452 4.381260833
```

Q updated: Q ← Q + A + B, where matrices A,B =

$$\begin{bmatrix} 0.156918 & 0.028530 \\ 0.028530 & 0.005187 \end{bmatrix} \qquad \begin{bmatrix} -0.961652 & -0.192033 \\ -0.192033 & -0.038347 \end{bmatrix}$$

Iteration 2

```
x = 1.567821549 2.648694827
F(x) = 0.3587612676
Improvement is 12.64123873
∇F(x) = ¯0.05985480199 0.3812608333

Q (approx. of Hessian inverse)     Actual Hessian inverse

[ 0.1952663236 ¯0.1635031489 ]    [ 0.8080949565  2.533897373 ]
[¯0.1635031489  0.96683987   ]    [ 2.533897373   8.445397811 ]

Search direction is 0.07402497392 ¯0.3784046231
Optimal stepsize is 0.196711875
Step is Δx= 0.01456159142 ¯0.07443668292,
  with magnitude 0.07584760846
Δgradient = 0.7795173394 ¯0.240617353

Q updated: Q ← Q + A + B, where matrices A,B =
[ 0.00724631285 ¯0.0370420702 ]  [¯0.155503537  0.2923208779 ]
[¯0.03704207027  0.189353537  ]  [ 0.292320877 ¯0.5495148012 ]
```


Iteration 3

```
x = 1.582383141 2.574258144
F(x) = 0.3441152698
Improvement is 0.01464599787
∇F(x) = 0.7196625374 0.1406434803

Q (approx. of Hessian inverse)     Actual Hessian inverse
[0.04700909863  0.09177565874]  [0.5818306937  1.841358161]
[0.09177565874  0.6066786058 ]  [1.841358161   6.32746822 ]

Search direction is ¯0.04673833525 ¯0.151372894
Optimal stepsize is 9.360119345
Step is Δx= ¯0.437476396 ¯1.416868353,
  with magnitude 1.482869356
Δgradient = 0.2727649394 ¯0.4474868063
Q updated: Q ← Q + A + B, where matrices A,B =
[0.3718379161 1.204282975]  [¯0.007777870003 ¯0.06786231974]
[1.204282975  3.900348571]  [¯0.06786231974  ¯0.5921022644 ]
```


Iteration 4

```
x = 1.144906745 1.157389791
F(x) = 0.04453617137
∇F(x) = 0.9924274768 ¯0.3068433261

Q (approx. of Hessian inverse)       Actual Hessian inverse
[0.411069144 1.228196314]    [0.3826013341 0.876085695]
[1.228196314 3.914924913]    [0.8760856959 2.506072844]
Search direction is ¯0.03109247191 ¯0.01762718775
Optimal stepsize is 3.5355748
Step is Δx= ¯0.1099297602 ¯0.0623222408,
  with magnitude 0.1263669809
Δgradient = ¯1.021376702 0.3546237095
Q updated: Q ← Q + A + B, where matrices A,B =
[0.1340066473 0.07597209829]  [¯0.00782901597 ¯0.0667960420]
[0.0759720982 0.04307069712]  [¯0.06679604207 ¯0.5698942564]
```


Iteration 5

```
x = 1.034976985 1.09506755
F(x) = 0.001794130712
Improvement is 0.04274204066
∇F(x) = ¯0.02894922502 0.0477803834
Q (approx. of Hessian inverse)   Actual Hessian inverse
[0.5372467761 1.237372371]  [0.5250889514 1.086909959]
[1.237372371  3.388101353]  [1.086909959  2.749853584]
Search direction is ¯0.04356924846 ¯0.1260638105
Optimal stepsize is 0.7330121416
Step is Δx= ¯0.03193678812 ¯0.09240630368,
  with magnitude 0.09776954226
Δgradient = 0.04878486626 ¯0.05463716156
Q updated: Q ← Q + A + B, where matrices A,B =
[0.2921858788 0.8454142898]  [¯0.357283507 ¯1.076685326]
[0.8454142898 2.44613232  ]  [¯1.076685326 ¯3.244625819]
```


Iteration 6

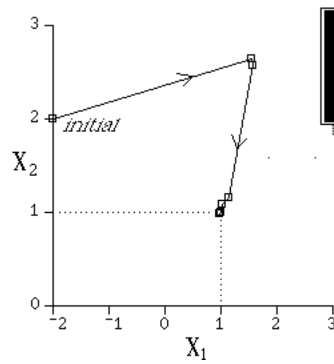
```
x = 1.003040197 1.002661247
F(x) = 0.00002099664646
Improvement is 0.001773134066
∇F(x) = 0.01983564124 ¯0.006856778164
Q (approx. of Hessian inverse)   Actual Hessian inverse
[0.4721491479 1.006101334]  [0.4965949585 0.9962094096]
[1.006101334  2.589607854]  [0.9962094096 2.498476164 ]
Search direction is ¯0.002466767447 ¯0.002200298527
Optimal stepsize is 1.242655471
Step is Δx= ¯0.003065342064 ¯0.002734213002,
  with magnitude 0.004107583561
Δgradient = ¯0.01979523147 0.006811426577
Q updated: Q ← Q + A + B, where matrices A,B =
[0.2234279599 0.1992924836]   [¯0.1836761831 ¯0.167746983 ]
[0.1992924836 0.1777642066]   [¯0.167746983  ¯0.1531992326]
```


Iteration 7

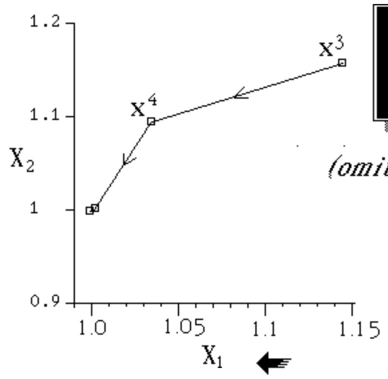
```
x = 0.9999748544 0.9999270337
F(x) = 1.146490964E¯9
Improvement is 0.00002099549997
∇F(x) = 0.00004040976709 ¯0.00004535158715
Q (approx. of Hessian inverse)   Actual Hessian inverse
[0.5119009247 1.037646835]  [0.4999773252 0.999929506]
[1.037646835  2.614172828]  [0.999929506  2.499808725]
Convergence criterion satisfied:
Gradient
                  *** CONVERGED ***
Solution found is 0.9999748544 0.9999270337
where F is 1.146490964E¯9
```



```
*** CONVERGED ***

Solution found is 0.9999748544 0.9999270337
where F is 1.146490964E¯9
and ∇F is 0.00004040976709 ¯0.00004535158715

# iterations = 7
# function evaluations= 60
# gradient evaluations= 34
Elapsed CPU time: 77.6 seconds
```

Davidon-Fletcher-Powell (DFP) Algorithm



Davidon-Fletcher-Powell (DFP) Algorithm

(omitting x^0, x^1, & x^2)

Powell's Method

In the example computation which follows, x= (3,3) was used as a starting point, rather than (-2, 2).
(My APL code terminated prematurely when (-2,2) was used to start the algorithm.)

Powell's Method

```
Tolerances:
  1-dimensional search: 0.001
  stopping criterion: |ΔF|≤ 0.00001
```

Cycle 1

```
Cycle Starts at x = 3 3
Where F(x) = 40, with Directions:
                         1) 1 0
                         2) 0 1
x = 1.67268 3
   F(x) = 0.49336, improvement = 39.5066 ← search in direction 1
x = 1.67268 2.79784
   F(x) = 0.452492, improvement = 0.0408677 ← search in second direction
New direction: ¯1.32732 ¯0.202158
x = 1.60911 2.78816    F(x) = 0.410587 ← search in new direction
```


Cycle 2

```
Cycle Starts at x = 1.60911 2.78816
Where F(x) = 0.410587, with Directions:
                  1)  0         1
                  2) ¯1.32732  ¯0.202158
Change in F during previous cycle was ΔF = 39.5894
x = 1.60911 2.58923
   F(x) = 0.371014, improvement = 0.0395727 ← search in direction 1
x = 1.54601 2.57962
   F(x) = 0.334028, improvement = 0.0369862 ← search in second direction
New direction: ¯0.063099 ¯0.208539
x = 1.1067 1.12771    F(x) = 0.0208058  ← search in new direction
```


Cycle 3

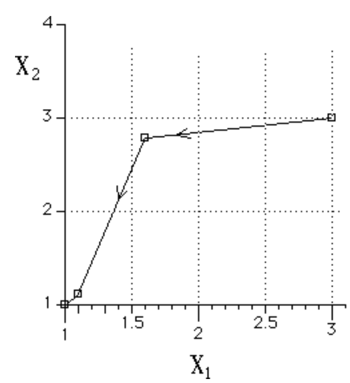
```
Cycle Starts at x = 1.1067 1.12771
Where F(x) = 0.0208058, with Directions:
                  1) ¯1.32732   ¯0.202158
                  2) ¯0.063099  ¯0.208539
Change in F during previous cycle was ΔF = 0.389781
x = 1.04624 1.11851
   F(x) = 0.00270882, improvement = 0.0180969
x = 1.01701 1.02192
   F(x) = 0.00044289, improvement = 0.00226593
New direction: ¯0.0896872 ¯0.105796
x = 1.00088 1.00289   F(x) = 2.05195E¯6
```


Cycle 4

```
Cycle Starts at x = 1.00088 1.00289
Where F(x) = 2.05195E¯6, with Directions:
                  1) ¯0.063099   ¯0.208539
                  2) ¯0.0896872  ¯0.105796
Change in F during previous cycle was ΔF = 0.0208037
x = 1.00001 1.00001
   F(x) = 9.0524E¯11, improvement = 2.05186E¯6
x = 1.00018 1.00021
   F(x) = 5.29529E¯8, improvement = ¯5.28624E¯8
New direction: ¯0.000701781 ¯0.00268185
x = 1.0002 1.00029   F(x) = 5.1093E¯8
```



```
*** CONVERGED ***

Solution found is 1.0002 1.00029
where F is 5.1093E¯8

# iterations = 4
# function evaluations= 123
Elapsed CPU time: 49.1 seconds
```

Powell's (version 1) method

Algorithm: Powell's (version 1) method

i	F	ΔF	logΔF
1	4E1	4E1	1.60206E0
2	4.10587E¯1	4.10587E¯1	¯3.86595E¯1
3	2.08058E¯2	2.08058E¯2	¯1.68182E0
4	2.05195E¯6	2.05195E¯6	¯5.68783E0
5	5.1093E¯8	5.1093E¯8	¯7.29164E0