Separable Programming

Stochastic LP with SIMPLE Recourse

Consider the 2-stage stochastic LP with ***simple*** recourse in which only the right-hand-side is random.

Cf. *Stochastic Programming*, by Willem K. Klein Haneveld and Maarten H. van der Vlerk, Dept. of Econometrics & OR, University of Groningen, Netherlands

P: $$\textit{Minimize}\;\; cx + E_\omega\left[\sum_{i=1}^{m_2}\tilde{Q}_i\left(y_i\right)\right]$$
$$\text{subject to}\;\; Ax \geq b$$
$$Tx - y(\omega) = h(\omega)$$
$$x \geq 0$$
$$y_i = y_i^+ - y_i^-,\; y_i^+ \geq 0,\; y_i^- \geq 0 \quad \forall i = 1, \ldots m_2$$

(The first-stage constraints might be instead "=" or "≤".)

The right-hand-side $h(\omega)$ may be interpreted as the random demand for a set of ***outputs***, with expected value h_ω.

The ***second-stage variables***

$$y(\omega) = Tx - h(\omega)$$

represent *surplus* (if positive) or *shortage* (if negative) of the outputs.

For example,

- y_i^+ = quantity of demand in excess of output (shortage of output) which must be acquired (at a cost q_i^+ per unit),
- y_i^- = shortage of demand (excess of output which must be disposed of) (at a cost q_i^- per unit),

where it is assumed that $q_i^+ + q_i^- > 0$.

Warning: the terminology & notation is confusing!

The expected second-stage cost is

$$Q_i(z) = E_\omega\left[\min_y \;\left\{q_i^+ y_i^+ + q_i^- y_i^- : y_i^+ - y_i^- = \omega - z, y_i^+ \geq 0, y_i^- \geq 0\right\}\right]$$
$$= q_i^+ G_i(z) + q_i^- H_i(z)$$

where $G_i(z)$ is the *expected surplus of demand (shortage of output)*:

$$G_i(z) = \int_{-\infty}^{+\infty} (t-z)^+ F_i(t)\, dt = \int_z^{+\infty}\left(1 - F_i(t)\right) dt$$

and $H_i(z)$ is the *expected shortage of demand (surplus of output)*:

$$H_i(z) = \int_{-\infty}^{+\infty} (z-t)^+ F_i(t)\, dt = \int_{-\infty}^{z} F_i(t)\, dt$$

If demand is random and a supply z is made available, $G_i(z)$ is

the expected demand in excess of the supply, i.e.,
the expected deficit in the supply.

Note the danger of confusion in the terminology!

The stochastic LP may therefore be restated as

$$\textit{Minimize}\;\; cx + \sum_{i=1}^{m_2} Q_i\left(z_i\right)$$
$$\text{subject to}\;\; Ax = b$$
$$Tx - z = 0$$
$$x \geq 0$$

If the probability distributions are *discrete*,
then $Q_i(z)$ is a ***piecewise-linear convex*** function.

This optimization problem can then be solved by an extension of LP usually called *"separable programming"*.

If the probability distributions are *continuous*, then a piecewise-linear *approximation* of each $Q_i\left(y_i\right)$ can be constructed.

Separable Programming

Suppose that for each output *i*, a set of J_i grid points is given,

$$\left\{\hat{z}_i^j\right\}_{j \in J_i}.$$

Represent each second-stage variable z_i as a *convex combination* of these grid points:

$$z_i = \sum_{j \in J_i} \lambda_i^j \hat{z}_i^j \quad \text{where} \quad \sum_{j \in J_i} \lambda_i^j = 1,\;\; \lambda_i^j \geq 0$$

and $Q_i(z_i)$ as the corresponding convex combination of function values:

$$Q_i\left(z_i\right) \approx \sum_{j \in J_i} \lambda_i^j \hat{q}_i^j\,, \quad \text{where}\;\; \hat{q}_i^j \equiv Q_i\left(\hat{z}_i^j\right)$$

The inner linearization of the original nonlinear problem P is the LP :

$$\textit{Minimize}\quad cx + \sum_{i=1}^{m_2}\sum_{j \in J_i} \hat{q}_i^j \lambda_i^j$$
$$\text{subject to}\quad Ax \geq b,$$
$$\sum_{j=1}^{n_1} T_{ij} x_j - \sum_{j \in J_i} \lambda_i^j \hat{z}_i^j = 0, \quad i = 1, 2, \ldots m_2$$
$$\sum_{j \in J_i} \lambda_i^j = 1, \quad i = 1, \ldots m_2$$
$$x \geq 0, \quad \lambda_i^j \geq 0 \;\; \forall i = 1, \ldots m_2 \;\&\; j \in J_i$$

Note that the variables of this problem are x and λ.

The m_2 convexity constraints are of type "**GUB**" (*Generalized Upper Bounds*), which are handled by many LP-solvers without increasing the size of the basis matrix.

Hence, when GUB facility is available, the *number of constraints* in the tableau is *identical* to that of the *expected value* problem (i.e., with random variables replaced by their expected values)!

The computational effort should therefore be of the same order of magnitude as that of the expected value problem!

This is sometimes referred to as the *"Lambda"* separable programming formulation, with the new variables associated with the *grid points* and convexity (GUB) constraints added.

An alternative formulation is
the *"Delta"* separable formulation, with
a new variable associated with each of the *intervals* between grid points, and
simple upper bounds (SUB) constraints added.

Computational efforts of the two formulations should be comparable, and results will be equivalent.

Example:

Stochastic Transportation Problem with Simple Recourse

Consider the small example with

- two sources, each with supply = 10, and
- three destinations, each with random demand.

Shipping Cost

	Dstn #1	Dstn #2	Dstn #3
Source #1	3	5	6
Source #2	2	4	7

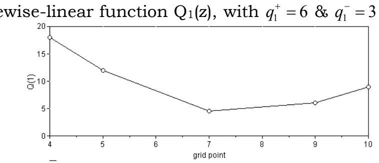
Surplus & Shortage Costs

	Dstn #1	Dstn #2	Dstn #3
q^+	6	7	8
q^-	3	3	6

Discrete Probability Distributions

```
Random demand #1, Mean = 7, # points = 3
          i:  1    2    3
          d:  5    7    9
          p:  0.25 0.5  0.25
--------------------
```

The piecewise-linear function $Q_1(z)$, with $q_1^+ = 6$ & $q_1^- = 3$:



$$\bar{Q}_1(z) = q_1^+ \sum_j p_j (d_j - z)^+ + q_1^- \sum_j p_j (z - d_j)^+$$

$$= \sum_j p_j \left[q_i^+ (d_j - z)^+ + q_i^- (z - d_j)^+ \right]$$

$$= 0.25\left[6(5-z)^+ + 3(z-5)^+\right]$$
$$+0.5\left[6(7-z)^+ + 3(z-7)^+\right]$$
$$+0.25\left[6(9-z)^+ + 3(z-9)^+\right]$$

That is,

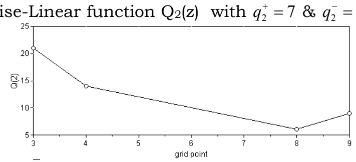
$$\bar{Q}_1(5) = 0.25[0+0] + 0.5[6(7-5)+0] + 0.25[6(9-5)+0] = 12$$
$$\bar{Q}_1(7) = 0.25[0+3(7-5)] + 0.5[0+0] + 0.25[6(9-7)+0] = 4.5$$
$$\bar{Q}_1(9) = 0.25[0+3(9-5)] + 0.5[0+3(9-7)] + 0.25[0+0] = 6$$

The piecewise-linear curve joins the points (5,12), (7,4.5), and (9,6), with slopes $-q_1^+ = -6$ on the left and $+q_1^- = +3$ on the right.


```
Random demand #2, Mean = 6, # points = 2

          i:  1   2
          d:  4   8
          p:  0.5 0.5
--------------------
```

The Piecewise-Linear function $Q_2(z)$ with $q_2^+ = 7$ & $q_2^- = 3$:




```
Random demand #3, Mean = 7, # points = 4
          i:  1   2   3   4
          d:  4   6   8   10
          p:  0.1 0.4 0.4 0.1
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```
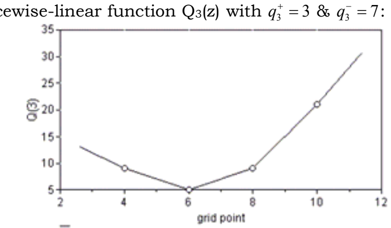

The piecewise-linear function $Q_3(z)$ with $q_3^+ = 3$ & $q_3^- = 7$:



Initial LP Tableau

First-Stage **Recourse**

rhs	-z	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	3	5	6	2	4	7	0	0	42	12	4.5	6	33	42	14	6	30	56	24	10.8	8.8	18	78
10	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	-5	-7	-9	-18	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	-4	-8	-16	0	0	0	0	0	0
0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	-4	-6	-8	-10	-20
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1

Compare the size of this tableau with that of the LP with the second-stage variables (y_k) for each scenario!


```
Solution of LP:  Objective: 96.3

First stage: nonzero variables
                    i   variable  value
                    1   X11         1
                    3   X13         6
                    4   X21         6
                    5   X22         4

Multipliers in convex combinations
          i   Grid #  Grid pt  Multipliers
          1     3       7          1
          2     7       4          1
          3    12       6          1

Second stage primal & dual solutions:
                 i  output  value   v      w
                 1  AAA       7     3    25.5
                 2  BBB       4     5    34.0
                 3  CCC       6     6    46.8
```

v & w are dual variables for 2nd-stage and convexity rows, respectively.

Optimal LP Tableau

		First-Stage								Recourse												
rhs	**-z**	1	2	**3**	**4**	**5**	6	**7**	8	1	2	**3**	4	5	6	**7**	8	9	10	11	**12**	13
¯96.3	1	0	0	0	0	0	2	0	1	16.5	1.5	0	7.5	61.5	8	0	12	76	9.2	1.2	0	10
3	0	0	0	0	0	0	0	1	1	¯7	¯2	0	2	11	¯4	0	4	12	¯6	¯2	0	2
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	¯1	0	1	0	1	0	1	0	0	0	0	0	¯4	0	4	12	0	0	0	0
4	0	0	1	0	0	1	0	0	0	0	0	0	0	0	4	0	¯4	¯12	0	0	0	0
6	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	6	2	0	¯2
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	0	0	0	¯1	0	¯1	7	2	0	¯2	¯11	4	0	¯4	¯12	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1