Generalized Linear Programming

Stochastic LP with SIMPLE Recourse

Consider the 2-stage stochastic LP with ***simple*** recourse in which only the right-hand-sides are random, and are continous random variables.

P: $$\text{Minimize } \; cx + \sum_{i=1}^{m_2} Q_i(z_i)$$
$$\text{subject to } \; Ax \ge b$$
$$Tx - z = 0$$
$$x \ge 0$$

(The first-stage constraints might be instead "=" or "≤".)

Here, The expected second-stage cost is

$$Q_i(z) = E_\omega\left[\min_y \; \left\{q_i^+ y_i^+ + q_i^- y_i^- : y_i^+ - y_i^- = \omega - z, y_i^+ \ge 0, y_i^- \ge 0\right\}\right]$$
$$= q_i^+ G_i(z) + q_i^- H_i(z)$$

where $G_i(z)$ is the *expected surplus*: $G_i(z) = \int_{-\infty}^{+\infty} (t-z)^+ F_i(t)\, dt$

and $H_i(z)$ is the *expected shortage*: $H_i(z) = \int_{-\infty}^{+\infty} (z-t)^+ F_i(t)\, dt$

$Q_i(z)$ is a finite, convex, continuous function of a single variable, and may be approximated by a piecewise-linear convex function as in *separable programming*:

Given, for each output i, a set of J_i grid points and the corresponding values of $Q_i(z)$:

$$\left\{\hat z_i^j\right\}_{j\in J_i} \text{ and } \left\{\hat q_i^j\right\}_{j\in J_i}, \text{ where } \hat q_i^j \equiv Q_i\left(\hat z_i^j\right)$$

Represent each variable z_i *and the function value* $Q_i(z)$ as a *convex combination* of these grid points and function values:

$$z_i = \sum_{j\in J_i} \lambda_i^j \hat z_i^j \text{ and } Q_i(z_i) \approx \sum_{j\in J_i} \lambda_i^j \hat q_i^j, \text{ where } \sum_{j\in J_i} \lambda_i^j = 1, \; \lambda_i^j \ge 0$$

The inner-linearization of the original nonlinear problem P is the LP :

$$\text{Minimize } \; cx + \sum_{i=1}^{m_2} \sum_{j\in J_i} \hat q_i^j \lambda_i^j$$
$$\text{subject to } \; Ax \ge b,$$
$$\sum_{j=1}^{n_1} T_{ij} x_j - \sum_{j\in J_i} \lambda_i^j \hat z_i^j = 0, \quad i = 1,2,\ldots m_2$$
$$\sum_{j\in J_i} \lambda_i^j = 1, \quad i = 1,\ldots m_2$$
$$x \ge 0, \quad \lambda_i^j \ge 0 \;\; \forall i = 1,\ldots m_2 \;\&\; j \in J_i$$

Note that the variables of this problem are x and λ.

Because the piecewise–linear approximation is an ***overestimate*** of $Q_i(z)$, the optimal solution of the approximating problem provides an ***upper*** bound on the solution of the exact problem!

Using a "finer" grid, with more grid points, improves the approximation, but increases the computational effort.

"Grid Refinement" is an iterative column-generating method for refining the grids, using dual information available after solving the current approximating separable programming LP

Grid Refinement

Suppose that we have solved the LP to get a primal solution $\left[\hat x, \hat z\right]$ and dual solution $\left[\hat u, \hat v, \hat w\right]$ where $\hat u, \hat v$, and $\hat w$ correspond to

- the first stage constraints, $Ax \ge b$
- the second stage constraints, $\sum_{j=1}^{n_1} T_{ij} x_j - \sum_{j\in J_i} \lambda_i^j \hat z_i^j = 0$ and
- the convexity constraints, $\sum_{j\in J_i} \lambda_i^j = 1$,

respectively.

What new grid points might be introduced in order to best improve the approximation?

A prospective grid point $\hat z_i$ for z_i would yield a new column for the LP, with elements of *zero* in all rows *except*:

- $-\hat z_i$ in row i of the second-stage constraints,
- 1 in row i of the set of convexity constraints

The ***reduced cost*** of this column would be

$$Q_i\left(\hat z_i\right) - [0, v, w]\begin{bmatrix} 0 \\ -ze_i \\ e_i \end{bmatrix} = Q_i\left(\hat z_i\right) + v_i \hat z_i - w_i$$

where e_i is the i^th unit vector.

It is reasonable to choose the grid point so as to *minimize* this reduced cost, i.e.,

$$\underset{z_i}{\text{Minimize}} \;\; Q_i(z_i) + v_i z_i - w_i$$

*This is an **un**constrained **one**-dimensional nonlinear minimization!*

For each $i=1, \ldots m_2$, find the optimal grid point.

If the minimum reduced cost is negative (or less than some tolerance), the column should be generated for the grid point and added to the LP tableau. *(The minimum reduced cost should never be positive, since a column for an existing grid point is basic and has zero reduced cost!)*

The sum of the m_2 minimum reduced costs provides a bound on the gap between the current approximate solution and the exact solution, and can be used in the *termination criterion* for the grid-refinement algorithm!

Minimizing the Reduced Cost function

$$\underset{z_i}{Minimize}\ \ Q_i(z_i)+v_i z_i - w_i$$

Case I: the random variable ω has *continuous* distribution F.

The optimal solution is achieved at a stationary point z such that

$$\frac{d}{dz}\left[Q_i(z)+v_i z - w_i\right]=\frac{d}{dz}Q(z)+v_i=0$$

Since

$$Q'(z)=-q^+ +\left(q^+ + q^-\right)F(z)$$

Therefore z must satisfy

$$-q^+ +\left(q^+ + q^-\right)F(z)+v=0 \Rightarrow \boxed{F(z)=\frac{q^+ - v}{q^+ + q^-}}$$

Note the similarity to the optimal solution of the Newsboy Problem:

q^+ ⟷ selling price, $-q^-$ ⟷ salvage value, v ⟷ acquisition cost

Case II: the random variable ω has *discrete* distribution

$$P\{\omega=\omega^s\}=p_s,\ \ s=1,\ldots S$$

The optimal solution is achieved at a point y such that 0 is a subdifferential of the reduced cost:

$$0\in\partial\left[Q(z)+vz-w\right]\Rightarrow -v\in\partial Q(z)$$

where $\partial Q(z)$ is the interval

$$\left[-q^+ +\left(q^+ + q^-\right)P\{\omega<z\}, -q^+ +\left(q^+ + q^-\right)P\{\omega\le z\}\right]$$

That is,

$$-q^+ +\left(q^+ + q^-\right)P\{\omega<z\}\le -v \le -q^+ +\left(q^+ + q^-\right)P\{\omega\le z\}$$

$$\Rightarrow \boxed{P\{\omega<z\}\le\frac{q^+ - v}{q^+ + q^-}\le P\{\omega\le z\}}$$

Example:

Stochastic Transportation Problem with Simple Recourse

Consider the small example with

- two sources, each with supply = 10, and
- three destinations, each with random demand.

Shipping Cost

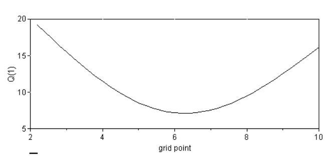
	Dstn #1	Dstn #2	Dstn #3
Source #1	3	5	6
Source #2	2	4	7

Surplus & Shortage Costs

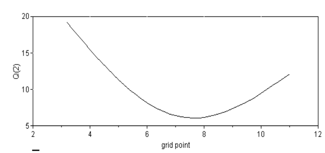
	Dstn #1	Dstn #2	Dstn #3
q^+	4	3	6
q^-	5	5	8

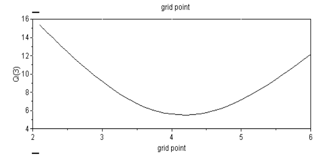
Demand Distribution

	Dstn #1	Dstn #2	Dstn #3
μ	6	7	4
σ	2	2	1



Expected 2nd-stage cost functions

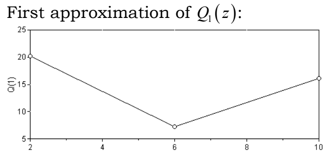




Expected 2nd-stage cost functions

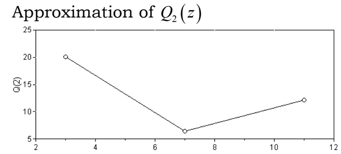
Initially, each $Q_i(z)$ is approximated by a three-point function, with grid points at μ and $\mu\pm 2\sigma$.

First approximation of $Q_1(z)$:



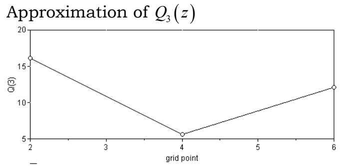
#	z	Q(z)	safety factor
1	6	7.181	0
2	2	20.156	−2
3	10	16.156	2

Approximation of $Q_2(z)$



#	z	Q(z)	safety factor
4	7	6.3831	0
5	3	20.1383	−2
6	11	12.1383	2

Approximation of $Q_3(z)$



#	z	Q(z)	safety factor
7	4	5.5852	0
8	2	16.1210	¯2
9	6	12.1210	2

The initial tableau is:

1st Stage **Recourse**

rhs	-z	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	9
0	1	3	5	6	2	4	7	0	0	7.181	16.16	20.16	6.383	12.14	20.14	5.585	12.12	16.12
10	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0	¯6	¯2	¯10	0	0	0	0	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	¯7	¯3	¯11	0	0	0
0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	¯4	¯2	0
1	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	¯6
1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1

iteration 1

Objective: 67.44

First stage: nonzero variables

i	variable	value
3	X13	2
4	X21	6
5	X22	3

Multipliers in convex combinations

i	Col #	Grid pt	Multipliers
1	1	6	1
2	5	3	1
3	8	2	1

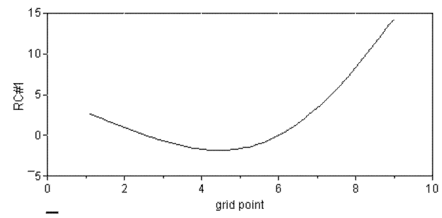
Second stage primal & dual solutions:

i	output	value	v	w
1	AAA	6	2	19.181
2	BBB	3	4	32.138
3	CCC	2	6	28.121

Refining the Grid:
For each random variable, solve a newsboy-type problem:

Demand #1: Using the dual variable $v_1 = 2$, we solve the "newsboy problem" by computing

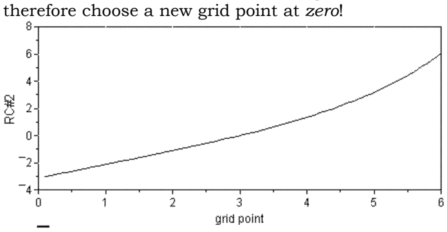
$$\hat{z} = F_1^{-1}\left(\frac{q_1^+ - v_1}{q_1^+ + q_1^-}\right) = F_1^{-1}\left(\frac{4-2}{4+5}\right) = F_1^{-1}(0.22222) = 4.47457$$



Demand #2: Using the dual variable $v_2 = 4$, we compute

$$\hat{z} = F_2^{-1}\left(\frac{q_2^+ - v_2}{q_2^+ + q_2^-}\right) = F_2^{-1}\left(\frac{3-4}{3+5}\right) = F_2^{-1}(-0.125)$$

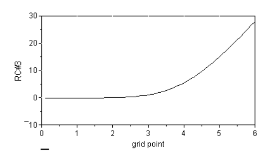
This indicates that there is no stationary point of the reduced cost function, but that it descends as one goes to the left-- we therefore choose a new grid point at *zero*!



Demand #3: Using the dual variable $v_2 = 4$, we compute

$$\hat{z} = F_3^{-1}\left(\frac{q_3^+ - v_3}{q_3^+ + q_3^-}\right) = F_3^{-1}\left(\frac{6-6}{6+8}\right) = F_3^{-1}(0)$$

Again, there is no stationary point minimizing the reduced cost function.



In this case, the new grid point z = 0.49206 was selected.

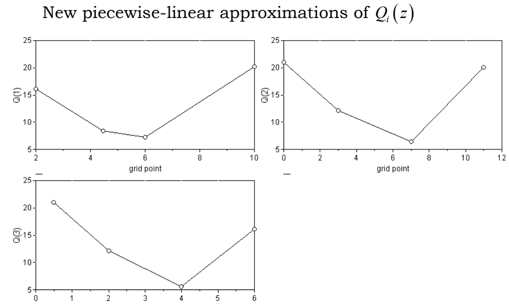
Refined Grid:

i	old z	p	grid pt	Q(z)	RC
1	6	0.22222	4.47457	8.397	¯1.83479
2	3	¯0.12500	0.00000	21.001	¯3.13732
3	2	0.00000	0.49206	21.048	¯0.12019

Sum of negative reduced costs: ¯5.0923 = bound on gap

The current LP optimum is 67.44 (an *upper* bound), and so 67.44-5.0923 = 62.35 is a *lower* bound on the optimum of the exact problem.

New piecewise-linear approximations of $Q_i(z)$



iteration 2

Solution of LP: Objective: 62.348

First stage: nonzero variables

i	variable	value
3	X13	0.49206
4	X21	4.47457

Multipliers in convex combinations

i	Col #	Grid pt	Multipliers
1	10	4.4746	1
2	5 11	3 0	0 1
3	12	0.49206	1

Second stage primal & dual solutions:

i	output	value	v	w
1	AAA	4.47457	2.0000	17.346
2	BBB	0.00000	2.9542	21.001
3	CCC	0.49206	6.0000	24.001

(v = dual variables for 2nd stage constraints,
w are for convexity constraints)

Refined Grid:

i	old z	p	grid pt	Q(z)	RC
1	4.47457	0.2222222	4.47453	8.3971	¯9.3110E¯10
2	0.00000	0.0057218	1.50000	16.5138	¯5.5814E¯2
3	0.49206	0.0000000	0.68571	19.8875	9.8568E¯4

Sum of negative reduced costs: ¯0.055814 = bound on gap

Currently upper bound = 62.348
lower bound = 62.348 -0.0558 = 62.29

iteration 3

Solution of LP: Objective: 62.348

First stage: nonzero variables

i	variable	value
3	X13	0.49206
4	X21	4.47457

Multipliers in convex combinations

i	Col #	Grid pt	Multipliers
1	10	4.4746	1
2	11 14	0 1.5	1 0
3	12	0.49206	1

Second stage primal & dual solutions:

i	output	value	v	w
1	AAA	4.47457	2.0000	17.346
2	BBB	0.00000	2.9914	21.001
3	CCC	0.49206	6.0000	24.001

Refined Grid:

i	old z	p	grid pt	Q(z)	RC
1	4.47457	0.2222222	4.47453	8.3971	¯9.3109E¯10
2	0.00000	0.0010706	0.73528	18.7981	¯3.3318E¯3
3	0.49206	0.0000000	0.68571	19.8875	9.8568E¯4

Sum of negative reduced costs: ¯0.0033318 = bound on gap

iteration 4

Solution of LP: Objective: 62.348

First stage: nonzero variables

i	variable	value
3	X13	0.49206
4	X21	4.47457

Multipliers in convex combinations

i	Col #	Grid pt	Multipliers
1	10	4.4746	1
2	11 16	0 0.73528	1 0
3	12	0.49206	1

Second stage primal & dual solutions:

i	output	value	v	w
1	AAA	4.47457	2.000	17.346
2	BBB	0.00000	2.996	21.001
3	CCC	0.49206	6.000	24.001