The Farmer's Problem

Stochastic LP with Recourse

Example problem in Birge & Louveaux, *Introduction to Stochastic Programming*

- A farmer raises **wheat**, **corn**, and **sugar beets** on 500 acres of land. Before the planting season he wants to decide how much land to devote to each crop.
- At least 200 tons of wheat and 240 tons of corn are needed for **cattle feed**, which can be purchased from a wholesaler if not raised on the farm.
- Any grain in excess of the cattle feed requirement can be sold at \$170 and \$150 per ton of wheat and corn, respectively.
- The wholesaler sells the grain for 40% more (namely \$238 and \$210 per ton, respectively.)
- Up to 6000 tons of sugar beets can be sold for \$36 per ton; any additional amounts can be sold for \$10/ton.

Crop yields are uncertain, depending upon weather conditions during the growing season.

Three **scenarios** have been identified ("good", "fair", and "bad"), each equally likely.

(In this data, only the yields are scenario-dependent, while in reality the purchase prices and sales revenues from grain would be higher in year with poor yield, etc.)

Scenario	Wheat yield (tons/acre)	Corn yield (tons/acre)	Beet yield (tons/acre)
1. Good	3	3.6	24
2. Fair	2.5	3	20
3. Bad	2	2.4	16

General Stochastic LP model:

$$Z = \min cx + \sum_{k=1}^{K} p_k q_k y_k \qquad (0.1)$$

subject to

$$T_k x + W y_k = h_k, k = 1, \ldots K; \qquad (0.2)$$

$$x \in X \qquad (0.3)$$

In this example, only T_k varies by scenario, while the cost vector q_k and the right-hand-side h_k are fixed.

Decision variables are

First stage: x_1 = acres of land planted in wheat

x_2 = acres of land planted in corn

x_3 = acres of land planted in beets

Second stage: w_1 = tons of wheat sold

w_2 = tons of corn sold

w_3 = tons of beets sold at \$36/T

w_4 = tons of beets sold at \$10/T

y_1 = tons of wheat purchased

y_2 = tons of corn purchased

The stochastic decision problem is

$$\text{Minimize } 150x_1 + 230x_2 + 260x_3 + \tfrac{1}{3}\sum_{k=1}^{3} Q_k(x)$$

$$\text{subject to } x_1 + x_2 + x_3 \le 500$$

$$x_j \ge 0, \text{ j=1,2,3}$$

where $Q_i(x)$ is the optimal solution of the second stage (recourse) problem after the scenario has been determined, given that the first stage variables x have been selected.

Recourse

$$Q_1(x) = \text{Minimum} -170w_1 - 150w_2 - 36w_3 - 10w_4 + 238y_1 + 210y_2$$

$$\text{s.t. } y_1 - w_1 \ge 200 - 3x_1$$

$$y_2 - w_2 \ge 240 - 3.6x_2$$

$$w_3 + w_4 \le 24x_3$$

$$y_1 \ge 0, y_2 \ge 0, \; w_1 \ge 0, w_2 \ge 0, \; 0 \le w_3 \le 6000, w_4 \ge 0$$

$$Q_2(x) = \text{Minimum} -170w_1 - 150w_2 - 36w_3 - 10w_4 + 238y_1 + 210y_2$$

$$\text{s.t. } y_1 - w_1 \ge 200 - 2.5x_1$$

$$y_2 - w_2 \ge 240 - 3x_2$$

$$w_3 + w_4 \le 20x_3$$

$$y_1 \ge 0, y_2 \ge 0, \; w_1 \ge 0, w_2 \ge 0, \; 0 \le w_3 \le 6000, w_4 \ge 0$$

$$Q_3(x) = \text{Minimum} \; -170w_1 - 150w_2 - 36w_3 - 10w_4 + 238y_1 + 210y_2$$

$$\text{s.t. } y_1 - w_1 \ge 200 - 2x_1$$

$$y_2 - w_2 \ge 240 - 2.4x_2$$

$$w_3 + w_4 \le 16x_3$$

$$y_1 \ge 0, y_2 \ge 0, \; w_1 \ge 0, w_2 \ge 0, \; 0 \le w_3 \le 6000, w_4 \ge 0$$

Solving Certainty Equivalent

All random parameters (in this case, T) are replaced by their expected values.

Tableau

b	z	X[1]	[2]	[3]	]	1	2	3	4	5	6	7	8	9	0
0	1	150	230	260	0	238	210	¯170	¯150	¯36	¯10	0	0	0	0
500	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
200	0	**2.5**	0	0	0	1	0	¯1	0	0	0	¯1	0	0	0
240	0	0	**3**	0	0	0	1	0	¯1	0	0	0	¯1	0	0
0	0	0	0	**¯20**	0	0	0	0	0	1	1	0	0	1	0
6000	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1

Solution

Optimal Solution

Found by solving certainty equivalent problem,
i.e., replacing all random parameters by their expected values.

Total cost: ¯118600

Stage One Variables:

i	variable	value	
1	X[1]	120	***Wheat acres***
2	X[2]	80	***Corn acres***
3	X[3]	300	***Beet acres***
4	slack 1	0	

Second Stage

i	variable	value	
1	Y[1]	0	
2	Y[2]	0	
3	W1	100	***Wheat sold***
4	W2	0	
5	W3	6000	***Beets sold***
6	W4	0	
7	surplus 1	0	
8	surplus 2	0	
9	slack 3	0	
10	slack 4	0	

Does this mean that the farmer's expected revenues will actually be 118600?

Evaluating this Trial Solution for Expected Cost:

First stage:

i	X[i]	
1	120	***Wheat acres***
2	80	***Corn acres***
3	300	***Beet acres***
4	0	

Second stage costs:

scenario k	cost	p[k]
1	¯29155.55556	0.3333333333
2	¯25888.88889	0.3333333333
3	¯18835.55556	0.3333333333

First stage cost: 114400.00
Expected second stage cost: ¯221640.00
Total: ¯107240.00

Using this planting plan, therefore, yields an expected **107240** revenue.

Tableau of Deterministic Equivalent LP

b	z	X[1]	X[2]	[3]	]	1	2	3	4	5	6	7	8	9	0	1	2
0	1	150	230	260	0	79.33	70	¯56.67	¯50	¯12	¯3.333	0	0	0	0	79.33	70
500	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
200	0	3	0	0	0	1	0	¯1	0	0	0	¯1	0	0	0	0	0
240	0	0	3.6	0	0	0	1	0	¯1	0	0	0	¯1	0	0	0	0
0	0	0	0	¯24	0	0	0	0	0	1	1	0	0	1	0	0	0
6000	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0
200	0	2.5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
240	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	¯20	0	0	0	0	0	0	0	0	0	0	0	0	0
6000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
200	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
240	0	0	2.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	¯16	0	0	0	0	0	0	0	0	0	0	0	0	0
6000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

continued….

(Tableau, continued)

3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0
¯56.67	¯50	¯12	¯3.333	0	0	0	0	79.33	70	¯56.67	¯50	¯12	¯3.333	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
¯1	0	0	0	¯1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	¯1	0	0	0	¯1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	¯1	0	0	0	¯1	0	0	0
0	0	0	0	0	0	0	0	0	1	0	¯1	0	0	0	¯1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1

"Picture" of LP Tableau

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Optimal Solution

(Found by solving deterministic equivalent problem directly, without decomposition)

Total cost: 108390

Stage One Variables:

i	variable	value	
1	X[1]	170	***Wheat Acres***
2	X[2]	80	***Corn Acres***
3	X[3]	250	***Beets Acres***
4	slack 1	0	

Second Stage

For each scenario, the optimal recourse variables are computed:

Scenario #1 "Good" yield

i	variable	value	
1	Y[1]	0	
2	Y[2]	0	
3	W1	310	***Sales of wheat***
4	W2	48	***Sales of corn***
5	W3	6000	***Sales of beets***
6	W4	0	
7	surplus 1	0	
8	surplus 2	0	
9	slack 3	0	
10	slack 4	0	

Scenario #2 "Fair" yield

i	variable	value	
1	Y[1]	0	
2	Y[2]	0	
3	W1	225	*Sales of wheat*
4	W2	0	
5	W3	5000	*Sales of beets*
6	W4	0	
7	surplus 1	0	
8	surplus 2	0	
9	slack 3	0	
10	slack 4	1000	

Scenario #3 "Bad" yield

i	variable	value	
1	Y[1]	0	
2	Y[2]	48	*Purchase of corn*
3	W1	140	*Sales of wheat*
4	W2	0	
5	W3	4000	*Sales of beets*
6	W4	0	
7	surplus 1	0	
8	surplus 2	0	
9	slack 3	0	
10	slack 4	2000	

Assuming "Perfect Information", i.e., *assuming that the farmer has advance knowledge of the quality of the yield and can base his decision upon that knowledge*

Solution for scenario #1 "Good" yield

Optimal cost: 167666.6667

Stage One Variables:

i	X[i]	
1	183.33	*Wheat Acres*
2	66.67	*Corn Acres*
3	250.00	*Beet Acres*
4	0.00	

Second-stage: nonzero variables

i	Y[i]	
3	350.00	*Sales of wheat*
5	6000.00	*Sales of Beets*

Solution for scenario #2 "Fair" yield

Optimal cost: 118600

Stage One Variables:

i	X[i]	
1	120.00	*Wheat Acres*
2	80.00	*Corn Acres*
3	300.00	*Beet Acres*
4	0.00	

Second-stage: nonzero variables

i	Y[i]	
3	100.00	*Sales of Wheat*
5	6000.00	*Sales of Beets*

Solution for scenario #3 "Bad" yield

Optimal cost: 59950

Stage One Variables:

i	X[i]	
1	100.00	*Wheat Acres*
2	25.00	*Corn Acres*
3	375.00	*Beet Acres*
4	0.00	

Second-stage: nonzero variables

i	Y[i]	
2	180.00	*Purchase of Corn*
5	6000.00	*Sales of Beets*

Expected value with perfect information:

$$\tfrac{1}{3}(167666.6667)+\tfrac{1}{3}(118600)+\tfrac{1}{3}(59950)=115405.56$$

What is the Value of Perfect Information (VPI) ?

(Expected value with perfect information) - *(Expected value without information)*

$$=115405-108390=7015$$