Relaxing the Exponential Assumption

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dennis-bricker@uiowa.edu

author

In queues which are modeled as birth-death processes,

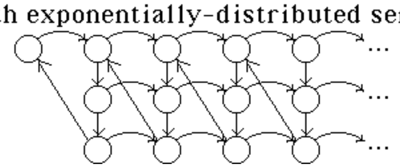
both the times between arrivals
and the service times

must have exponential distributions.

Exponential distributions have coefficient of variation equal to 1, i.e., $\frac{\sqrt{\mathrm{var}[T]}}{E[T]} = 1$

If, in an application, inter-arrival &/or service times are either more or less *regular*, what can be done?

We have seen that an $M/E_k/1$ queue, for which service times have Erlang-k distribution, can be modeled as a (continuous-time) Markov chain.
That is, the service consists of k phases, each with exponentially-distributed service time.



If service time T has Erlang-k dist'n with mean $1/k\mu$,
then

$$T = \sum_{i=1}^{k} Y_i\ ,\quad E[T] = \sum_{i=1}^{k} E[Y_i] = kE[Y_i] = 1/\mu$$

$$\mathrm{Var}[T] = \sum_{i=1}^{k} \mathrm{Var}[Y_i] = k \times \mathrm{Var}[Y_i] = k/(k\mu)^2 = 1/k\mu^2$$

$\Rightarrow$ **Coefficient of variation** $= 1/\sqrt{k}$

The coefficient of variation may be made as small as we like (but >0) by increasing k.

An Erlang-k random variable is a *convolution* of k random variables with exponential dist'n, and is *more regular* than a random variable with exponential distribution.

To approximate distributions which are *less regular*, i.e., have c.v. > 1, we can use a *hyper-exponential* distribution.

$$P\{T \le t\} = F(t) = \beta\left[1-e^{-\mu_1 t}\right] + (1-\beta)\left[1-e^{-\mu_2 t}\right]$$

where $0 < \beta < 1$.

Hyper-Exponential Distribution

A hyper-exponential dist'n is a *mixture* of exponential distributions, with service rate μ defined by

$$\frac{1}{\mu} = \frac{\beta}{\mu_1} + \frac{(1-\beta)}{\mu_2}$$

and has coefficient of variation >1 and can be made arbitrarily large.

M/H/1 Queue

Arrivals are Poisson with rate λ , and service time has hyper-exponential dist'n

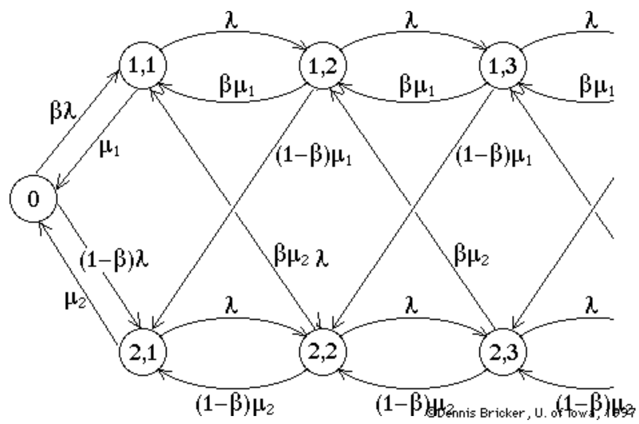
$$F(t) = \beta\left[1-e^{-\mu_1 t}\right] + (1-\beta)\left[1-e^{-\mu_2 t}\right]$$

Equivalently, suppose 2 types of customers,
Type 1 with service time distn Exp(μ_1)
Type 2 with service time distn Exp(μ_2)
where β = fraction of customers that are type 1

States

(i,j)

where $\begin{cases} \text{i=type of service being provided} \\ \text{j=\# in system} \end{cases}$



Phase-type Distribution

A distribution function H is phase-type with k phases

if it is $(1-\beta_1)F_1 + (1-\beta_2)\beta_1 F_1 \oplus F_2 + \cdots$
$+\beta_1 \beta_2 \cdots \beta_{k-1} F_1 \oplus F_2 \oplus \cdots \oplus F_k$

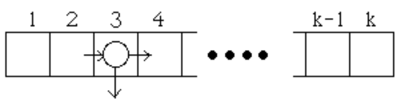
where F_i is exponential dist'n with rate μ_i

$F_1 \oplus F_2$ is the convolution of F_1 and F_2,

$0 < \beta_i < 1$ for $i=1,2,\ldots k-1$, $\beta_k = 0$

©Dennis Bricker, U. of Iowa, 1997

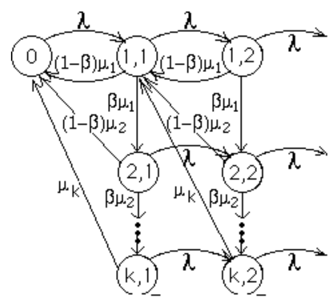
Phase-type Distribution



Think of the service facility as having k stages, with a service time in stage j having exponential dist'n (rate μ_j),

and upon completion of stage j, the service is complete with probability $(1-\beta_j)$ or customer enters stage j+1.

©Dennis Bricker, U. of Iowa, 1997



States (i,j)

i=phase of service in progress

j= # of customers in the system

©Dennis Bricker, U. of Iowa, 1997

Phase-type Distribution

The Erlang & Hyper-exponential dist'ns are both phase-type distributions.

Any arbitrary distribution can be approximated as closely as desired by a phase-type dist'n.

©Dennis Bricker, U. of Iowa, 1997

example Suppose that service time T is discrete, with $P\{T=a\} = 1-\beta$
and $P\{T=b\} = \beta$, for $\beta \in (0,1)$.

That is, $T = a + (b-a) I$
where $P\{I=1\} = \beta$, $P\{I=0\} = 1-\beta$.

Consider the distribution $(1-\beta) E_{k'} + \beta E_{k'} \oplus E_{k''}$

where $E_{k'}$ is Erlang-k' with phase rate k'/a

$E_{k''}$ is Erlang-k" with phase rate k"/(b−a)

As $k' \to \infty$ & $k'' \to \infty$, this dist'n converges to that of T.

©Dennis Bricker, U. of Iowa, 1997

Using phase-type distributions, we can, in principle, approximate any queue as a continuous-time Markov chain....

In practice, the state space of this Markov chain may be large &/or complex, and the balance equations intractable.

As the distributions become more regular, i.e., coefficient of variation decreases, the performance measures of the queueing system usually improve.

©Dennis Bricker, U. of Iowa, 1997