Jackson Network of Queues

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Jackson Network of Queues

a collection of queues with *exponential* service times in which customers travel from one queue to another according to a Markov chain--

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Jackson Network of Queues

- the network consists of N service centers, where service center i contains c_i identical servers and a queue with *infinite* capacity
- customers from outside the network (called *exogenous* customers) arrive at service center i according to a Poisson process with rate λ_i. *(Arrival processes are independent.)*

Jackson Network of Queues

- after receiving service at center i, a customer leaves the network with probability $p_{io} \geq 0$ or goes instantaneously to service center j with probability p_{ij}

(independent of number of customers at that center or number in the system)

Jackson Network of Queues

- customers arriving at center i are served FIFO (first-in-first-out), and service times are exponentially distributed with mean $1/\mu_i(s_i)$ where s_i = # of customers at center i.

(Service rate at each center may depend only on the number of customers at that center.)

Let $X_i(t)$ = # of customers at service center i at time t

State of system: $s=(s_1,s_2, \ldots s_N)$

$P(s;t) = P(s_1,s_2, \ldots s_N;t) = P\{X_i(t)=s_i\ ,\ i=1,2,\ldots N\}$

Steady-state distribution

$$\pi_s = \lim_{t\to\infty} P(s;t)$$

Jackson Networks of queues have the very nice property that the steady-state distribution has a ***product*** form:

$$\pi_s = \pi^1_{s_1} \times \pi^2_{s_2} \times \cdots \times \pi^N_{s_N}$$

☞ Open Jackson Networks

$\lambda_i > 0$ for some i

$p_{jo} \neq 0$ for some j

At one or more service centers, customers may arrive from outside network &/or depart the network

☞ Closed Jackson Networks

$\lambda_i = 0$ & $p_{io} = 0\ \forall i$

customers circulate among service centers, but no exogenous arrivals or departures

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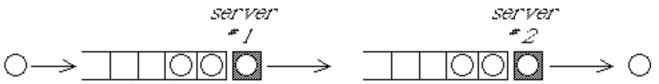
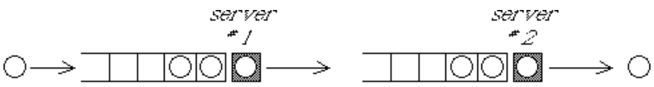
- If $\lambda_i > 0$ for some i, the network is *open*.

Open Jackson Networks

Customers may arrive from outside the system, and may depart the system.
The total number of customers in the network fluctuates.



example: tandem queues



Recall that for the two infinite-capacity tandem queues, the balance equations were satisfied by

$$\pi_{s_1,s_2} = \pi^1_{s_1} \times \pi^2_{s_2}$$

(product-form distribution)

where

$$\pi^i_{s_i} = (1-\rho_i)\,\rho_i^{s_i}, \qquad \rho_i = \lambda/\mu_i$$

is the steady-state distribution for the M/M/1 queue!

In the case of tandem queues, we know the average arrival rate at the second queue to be λ.

More generally, when arrivals at a service center may be exogenous or from any of the other centers, we must compute the composite arrival rate of each center by solving "traffic equations".

Traffic Equations

Let λ_i = exogenous arrival rate at service center i

α_i = *departure* rate in steady state at service center i

$$\left.\begin{matrix}\text{average rate}\\ \text{of departures}\end{matrix}\right\} = \left\{\begin{matrix}\text{average rate}\\ \text{of arrivals}\end{matrix}\right.$$

Then

$$\boxed{\alpha_i = \lambda_i + \sum_{j=1}^{N} \alpha_j\, p_{ji}} \quad \text{for } i=1,2,\ldots N$$

Given λ_i and p_{ij}, this system of linear equations has a unique, nonnegative solution

Traffic Equations

Since, in steady state, the composite rate of arrivals (external & internal) must equal the rate of departure of each center,

α_i = *composite arrival rate* in steady state at service center i

$$\boxed{\alpha_i = \lambda_i + \sum_{j=1}^{N} \alpha_j\, p_{ji}} \quad \text{for } i=1,2,\ldots N$$

$\rho_i^{n-c_i}$

Jackson's Theorem

Consider an open Jackson network, for which $\alpha_i < c_i\mu_i$

Then the limiting probabilities exist,

$$\boxed{\rho_i \equiv \frac{\alpha_i}{c_i\mu_i}}$$

and

$$\pi_s = \prod_{i=1}^{N} \Psi_i(s_i)$$

where

$$\Psi_i(n) = \begin{cases} \Psi_i(0)\,\dfrac{(c_i\rho_i)^n}{n!} & \text{if } n \le c_i \\ \Psi_i(0)\,\dfrac{(c_i\rho_i)^n}{c_i!\;c_i^{n-c_i}} & \text{if } n \ge c_k \end{cases}$$

and $\Psi_i(0)$ is a normalizing constant which is chosen to yield $\sum_{n=0}^{\infty} \Psi_i(n) = 1$ for each i.

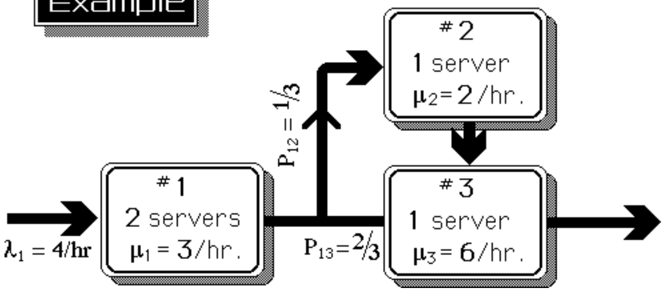
Compare

$$\Psi_i(n) = \begin{cases} \Psi_i(0)\,\dfrac{(c_i\rho_i)^n}{n!} & \text{if } n \le c_i \\ \Psi_i(0)\,\dfrac{(c_i\rho_i)^n}{c_i!\;c_i^{n-c_i}} & \text{if } n \ge c_k \end{cases}$$

with the steady-state distribution for the M/M/c queue, with infinite capacity:

$$\pi_n = \begin{cases} \pi_0\,\dfrac{(c\rho)^n}{n!} & ,\ n=1,\,2,\,\ldots c \\ \pi_0\,\dfrac{(c\rho)^n}{c!\;c^{n-c}} & ,\ n=c+1,\,c+2,\,\ldots \end{cases}$$

Example



Traffic equations

1	2	3	ω
1	0	0	4
¯0.33333	1	0	0
¯0.66667	¯1	1	0

(ω=exogenous arrival rates)

$$\alpha_i = \lambda_i + \sum_j \alpha_j\, p_{ij} \quad \forall i$$

i.e.,

$$\begin{cases} \alpha_1 = \lambda_1 \\ \alpha_2 = 0 + \ \alpha_1\, p_{12} \\ \alpha_3 = 0 + \ \alpha_1\, p_{13} + \alpha_2 \end{cases}$$

Solution of Traffic equations: Net Arrival Rates:

node	1	2	3
rate	4	1.3333	4
min c	2	1	1

i.e.,

$$\begin{cases} \alpha_1 = 4/\text{hr} \\ \alpha_2 = 4/3\ /\text{hr} \\ \alpha_3 = 4/\text{hr} \end{cases}$$

Steady-State Distribution

i	1	2	3
0	0.200000	0.333333	0.333333
1	0.266667	0.222222	0.222222
2	0.177778	0.148148	0.148148
3	0.118519	0.098765	0.098765
4	0.079012	0.065844	0.065844
5	0.052675	0.043896	0.043896
6	0.035117	0.029264	0.029264
7	0.023411	0.019509	0.019509
8	0.015607	0.013006	0.013006
⋮	⋮	⋮	⋮

For example,

$$\pi_{0,0,0} = \pi_0^1 \times \pi_0^2 \times \pi_0^3$$

$$= \frac{1}{5} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{45}$$

$$= 0.022222$$

$$\pi_{1,1,1} = \pi_1^1 \times \pi_1^2 \times \pi_1^3$$

$$= \frac{4}{15} \times \frac{2}{9} \times \frac{2}{9} = \frac{16}{1215}$$

$$= 0.0131687$$

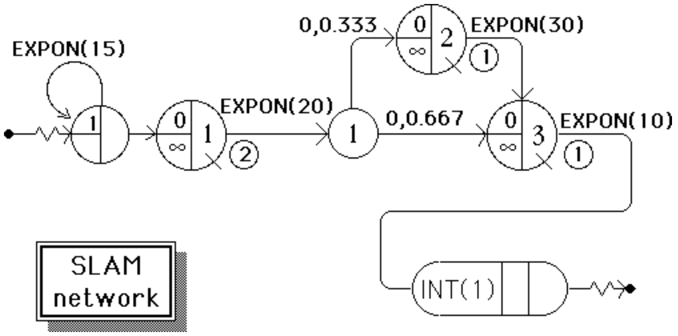
Expected number of visits
to nodes of a Jackson network,
beginning at any node,
before unit exits the network

from \ to	1	2	3
1	1	0.33333	1
2	0	1	1
3	0	0	1

i	Lq	Wq	L	W
1	1.066667	0.266667	2.400000	0.600000
2	1.333333	1.000000	2.000000	1.500000
3	1.333333	0.333333	2.000000	0.500000

Lq=length of queue
Wq=waiting time
L=# at node
W=time at node
(times are time/visit to node) (hours)

Totals: Sum of Lq= 3.7333, Sum of L (L_total) = 6.4
Average total time in system (by Little's Law):
Wtotal = L_total ÷ sum of exogenous arrival rates (4)
Wtotal = 1.6



SLAM network

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                    *** FILE STATISTICS ***

 FILE              AVERAGE  STD   MAX   CURRENT  AVERAGE
 NUMBER             LENGTH  DEV. LENGTH LENGTH  WAIT TIME
 1       QUEUE      1.079   2.254  10      3      15.892
 2       QUEUE      2.529   2.577   9      0     112.391
 3       QUEUE      1.182   1.918   9      7      17.672
                   *** SERVICE ACTIVITY STATISTICS ***

ACT  ACT LABEL OR  SER  AVG   STD  CUR  MAX IDL  MAX BSY  ENT
NUM  START  NODE   CAP  UTIL  DEV  UTIL TME/SER  TME/SER  CNT
 1          QUEUE   2  1.298  0.79  2    2.00      2.00   321
 2    Q2    QUEUE   1  0.775  0.42  0  214.25   1000.71   108
 3    Q3    QUEUE   1  0.659  0.47  1  164.87    372.59   313
```

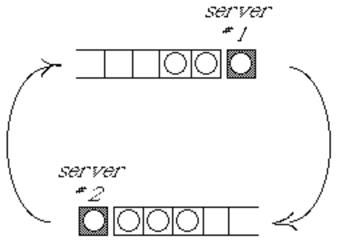
```
  ** STATISTICS FOR VARIABLES BASED ON OBSERVATION **


  MEAN    STANDARD   COEFF OF  MINIMUM   MAXIMUM   NO. OF
  VALUE   DEVIATION  VARIATION   VALUE     VALUE     OBS
0.112E+03 0.105E+03 0.935E+00 0.526E+01 0.483E+03  313
```

Average time in system
112 minutes = 1.86667 hours

- If $\lambda_i = 0$ & $p_{i0} = 0\ \forall i$ the network is *closed*.

Closed Jackson Networks



No exogenous arrivals or departures from the system... the total number of customers in the system remains constant!

Traffic Equations

Let α_i = *departure* rate in steady state at service center i

average rate of departures } = { *average rate of arrivals*

Then $\boxed{\alpha_i = \sum_{j=1}^{N} \alpha_j p_{ji}}$ for i=1,2,...N

Because the system of equations is homogeneous, the solution is not unique! Any multiple of a solution is also a solution.

Jackson's Theorem for Closed Networks

Let M=# of customers in the system

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ be any nonnegative, nonzero solution of the traffic equations, and let $\rho_i \equiv \frac{\alpha_i}{c_i \mu_i}$

The possible states of the system are elements of

$$S=\left\{ s \;\middle|\; \sum_{i=1}^{N} s_i = M \right\}$$

Then the steadystate probabilities are given by

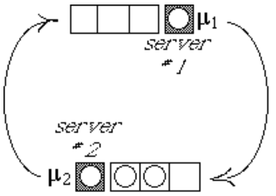
$$\pi_s = K \prod_{i=1}^{N} \Psi_i(s_i) \quad \text{for } s \in S$$

product form of joint dist'n

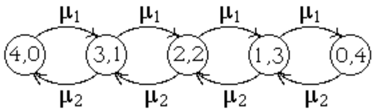
where

$$\Psi_i(n) = \begin{cases} \frac{(c_i\rho_i)^n}{n!} & \text{if } n \le c_i \\ \frac{(c_i\rho_i)^n}{c_i!\, c_i^{n-c_i}} & \text{if } n \ge c_k \end{cases}$$

and K is a "normalizing constant" such that $\sum_{s\in S} \pi_s = 1$



Recall 2 cyclic queues with 4 customers:



Transition diagram is equivalent to that of M/M/1/4 queue, with

Is this of the product form?

$$\pi_{s_2} = \rho^{s_2}\left[\frac{1-\rho}{1-\rho^5}\right], \quad \rho = \frac{\mu_1}{\mu_2}$$

Closed Jackson Networks

The steady-state distribution for this cyclic network of 2 queues & 4 customers is also of the product form:

```
4 = M = # units in system
2 = N = # nodes in system

 i   n   μ

 1   1   1
 2   1   2
```

Let

μ_1 = 1/hr.

μ_2 = 2/hr.

```
Traffic equations

 1   2   b

 1  ⁻1   0
⁻1   1   0
 1   1   1

(Solution is not unique; last row normalizes α)
Solution of Traffic equations: Arrival Rates:

             node 1   2
             rate 0.5 0.5
```

```
              Ψ1        Ψ2
PSI    i      1          2

       0 1.000000 1.000000
       1 0.500000 0.250000
       2 0.250000 0.062500
       3 0.125000 0.015625
       4 0.062500 0.003906

Normlizing constant K: 8.2581
# of states = 5
```

$$\Psi_i(n) = \begin{cases} \frac{(c_i\rho_i)^n}{n!} & \text{if } n \le c_i \\ \frac{(c_i\rho_i)^n}{c_i!\, c_i^{n-c_i}} & \text{if } n \ge c_k \end{cases}$$

Calculating the Normalizing Constant K

	Ψ_1	Ψ_2
i	1	2
0	1.000000	1.000000
1	0.500000	0.250000
2	0.250000	0.062500
3	0.125000	0.015625
4	0.062500	0.003906

$$\sum_{s\in S} \Psi_1(s_1)\times\Psi_2(s_2) = (1.0)(0.003906)+(0.5)(0.015625) +(0.25)(0.0625)+(0.125)(0.25)+(0.0625)(1.0)$$

$$= 0.1210935$$

So, in order that the probabilities will sum to 1.0,

$$K = {}^{1}/_{0.1210935} = 8.2580816$$

For large values of M (# customers) and N (# of service centers), the number of elements of the state set S will be extremely large, making the computation of K by enumerating the possible states very burdensome.

There are, however, recursive methods of computing K which avoid much of the computational burden.

Steady-State Distribution

Once K is found, then the probability of each state may be computed:

#	1 2	PI
1	0 4	0.032258
2	1 3	0.064516
3	2 2	0.12903
4	3 1	0.25806
5	4 0	0.51613

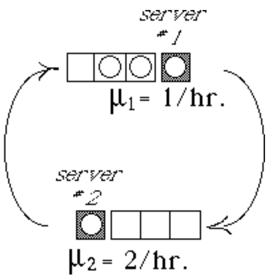
$$\pi_{0,4} = K\ \Psi_1(0)\times\Psi_2(4)$$
$$= 8.2580816\times 1.0\times 0.003906$$

Average Numbers at Nodes

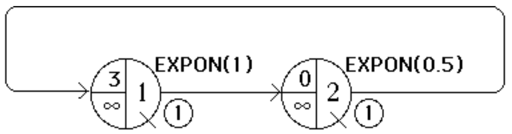
i	L
1	3.16129032
2	0.83870968

Unlike the case of the open Jackson Network, we do not know the average arrival rates at the service centers, and so we cannot use Little's Formula to compute the average waiting time at each service center!

Let's try forming a SLAM model of the 2 cyclic queues:



SLAM network



3 customers initially in queue #1 implies that server #1 is busy, i.e., that there are initially 4 customers in the network.

```
                  *** FILE STATISTICS ***

 FILE                 AVERAGE  STD   MAX   CURRENT  AVERAGE
 NUMBER                LENGTH  DEV. LENGTH LENGTH  WAIT TIME

   1   Q1  QUEUE       2.178  1.005   3       3     2.204
   2   Q2  QUEUE       0.363  0.749   3       0     0.368
```

Lq

```
                  *** SERVICE ACTIVITY STATISTICS ***

ACT  ACT LABEL OR  SER  AVG   STD  CUR  MAX IDL  MAX BSY  ENT
NUM  START NODE    CAP  UTIL  DEV  UTIL TME/SER  TME/SER  CNT

 1    Q1   QUEUE    1  0.968 0.10  1    3.00    191.49   4740
 2    Q2   QUEUE    1  0.491 0.50  0   10.74     12.10   4740
```