Duality Theory for Quadratic Programming

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QP: Minimize $\frac{1}{2}x^T Q x + c^T x$ subject to $Ax \geq b$

i.e., Minimize $f(x)$ s.t. $g(x) \leq 0$, $x \in X$

where
$$\begin{cases} f(x) = \frac{1}{2}x^T Q x + c^T x \\ g(x) = b - Ax \leq 0 \\ X = R^n \end{cases}$$

Assume that Q is positive semidefinite, so that f(x) is convex.

Lagrangian Function

$$L(x,\lambda) = f(x) + \lambda^T g(x)$$
$$= \frac{1}{2}x^T Q x + c^T x + \lambda^T(b - Ax)$$

Dual Objective Function

$$\widehat{L}(\lambda) = \min_x \left\{ \frac{1}{2}x^T Q x + c^T x + \lambda^T(b - Ax) \right\}$$

For each value of λ, an unconstrained minimization of a convex quadratic function must be performed!

Because of the convexity of the Lagrangian function, the optimal x must be a stationary point of the Lagrangian function:

$$\nabla_x L(\bar{x}(\lambda), \lambda) = 0 \Leftrightarrow \widehat{L}(\lambda) = L(\bar{x}(\lambda), \lambda)$$

i.e., for each λ, we must choose x to satisfy

$$\nabla_x L(x, \lambda) = Qx + c - A^T\lambda = 0$$
$$\Rightarrow x^T\left(Qx + c - A^T\lambda\right) = x^T(0)$$
$$\Rightarrow x^T Qx + x^T c - x^T A^T \lambda = 0$$
$$\Rightarrow \boxed{x^T Qx + c^T x - \lambda^T A x = 0}$$

Dual Objective Function

$$\widehat{L}(\lambda) = \min_x \left\{ \frac{1}{2}x^T Q x + c^T x + \lambda^T(b - Ax) \right\}$$

$$= \frac{1}{2}x^T Q x + c^T x + \lambda^T(b - Ax)$$

where x is chosen to satisfy

$$x^T Qx + c^T x - \lambda^T A x = 0$$

Dual Objective Function

$$\widehat{L}(\lambda) = \frac{1}{2}x^T Q x + c^T x + \lambda^T(b - Ax)$$

where $x^T Qx + c^T x - \lambda^T A x = 0$

$$= \lambda^T b - \frac{1}{2}x^T Q x + \underbrace{x^T Q x + c^T x - \lambda^T A x}_{=0}$$

Therefore,

$$\widehat{L}(\lambda) = \lambda^T b - \frac{1}{2}x^T Qx$$ *where x must satisfy* $Qx + c - A^T\lambda = 0$

LAGRANGIAN DUAL OF QP

$$\text{Maximize}_{\lambda \geq 0} \ \widehat{L}(\lambda)$$

$$\text{Maximize } \lambda^T b - \frac{1}{2}x^T Q x$$
$$\text{subject to } Qx + c - A^T\lambda = 0$$
$$\lambda \geq 0$$

That is, the Lagrangian dual of the quadratic programming problem QP is another quadratic programming problem with only nonnegativity constraints!

If Q is positive definite, i.e., f(x) is strictly convex, then Q is nonsingular, and

$$Qx + c - A^T\lambda = 0$$

can be solved by inverting Q:

$$\boxed{\bar{x}(\lambda) = Q^{-1}\left[A^T\lambda - c\right]}$$

$$\boxed{\bar{x}(\lambda) = Q^{-1}\left[A^T\lambda - c\right]}$$

This can be used to eliminate x from the statement of the Dual Problem:

$$\text{Maximize } \lambda^T b - \frac{1}{2}x^TQx = b^T\lambda - \frac{1}{2}\left[Q^{-1}(A^T\lambda - c)\right]^T Q\left[Q^{-1}(A^T\lambda - c)\right]$$

subject to $\lambda \geq 0$

So the dual objective, expressed in terms of λ, is

$$b^T\lambda - \frac{1}{2}\left[Q^{-1}(A^T\lambda - c)\right]^T Q\left[Q^{-1}(A^T\lambda - c)\right]$$

$$= b^T\lambda - \frac{1}{2}\left[(A^T\lambda - c)^T Q^{-1}(A^T\lambda - c)\right]$$

$$= b^T\lambda - \frac{1}{2}\left[\lambda^T A Q^{-1}A^T\lambda - 2c^TQ^{-1}A^T\lambda + c^TQ^{-1}c\right]$$

$$= \left[b^T + c^TQ^{-1}A^T\right]\lambda - \frac{1}{2}\lambda^T\left[AQ^{-1}A^T\right]\lambda - \frac{1}{2}c^TQ^{-1}c$$

Thus the dual problem can be written as

$$\boxed{\begin{array}{c}\text{Maximize } e^T\lambda + \frac{1}{2}\lambda^T D\lambda \\ \text{subject to } \lambda \geq 0\end{array}} \quad -\frac{1}{2}c^TQ^{-1}c$$

constant

where $\begin{cases} e = b + AQ^{-1}c \\ D = -AQ^{-1}A^T \end{cases}$

Compare the sizes of the two problems:

PRIMAL:	DUAL:
n variables	m variables
m constraints (inequalities)	m constraints (nonnegativity)

It would appear that the Dual QP problem is more computationally attractive...

especially if the number of primal variables is more than the number of constraints!

However,

Note that in QP we included included no explicit nonnegativity constraints... if $x \geq 0$ is to be included, we must include in the constraints $\begin{bmatrix} A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ 0 \end{bmatrix}$

This adds n primal constraints ⇒ # of dual variables will be m+n.

EXAMPLE

$$\text{Minimize } \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - 2x_1 - 2x_2$$

subject to $\begin{cases} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \end{cases}$

that is, $\text{Minimize } \frac{1}{2}x^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} -2 \\ -2 \end{bmatrix}^T x$

subject to $\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

To write the dual QP, we must compute

$$\begin{cases} e = b + AQ^{-1}c \\ D = -AQ^{-1}A^T \end{cases}$$

using $\begin{cases} Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \\ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \end{cases}$

$$D = -AQ^{-1}A^T = -\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\mathbf{e} = \mathbf{b} + AQ^{-1}\mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}$$

Dual QP Problem

$$\text{Maximize } \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}^T \lambda + \frac{1}{2}\lambda^T \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \lambda$$

subject to $\lambda \geq 0$

$$\begin{cases} \text{Maximize } \lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4 \\ \qquad - \frac{1}{2}\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2\right] + \lambda_1\lambda_3 + \lambda_2\lambda_4 \\ \text{subject to } \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0,\ \lambda_4 \geq 0 \end{cases}$$

After finding the optimal dual solution ,
we can compute the optimal primal solution:

$$x^*(\lambda^*) = Q^{-1}\left[A^T\lambda^* - c\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \lambda^* - \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right)$$

$$\Longrightarrow \begin{cases} x_1^* = -\lambda_1^* + \lambda_3^* + 2 \\ x_2^* = -\lambda_2^* + \lambda_4^* + 2 \end{cases}$$