Complementary Pivoting Algorithm for Quadratic Programming

author

Consider the QP problem:

QP: Minimize $\frac{1}{2}x^T Q x + c^T x$
subject to $Ax \geq b$

Karush-Kuhn-Tucker conditions:

$$
\begin{aligned}
Qx - A^T\lambda &= -c \\
Ax - y &= b \\
\lambda_i y_i &= 0 \ \forall\, i=1,\cdots m \\
\lambda \geq 0,&\quad y \geq 0
\end{aligned}
$$

(X unrestricted in sign!)

λ = *Lagrangian multipliers*
y = *primal surplus variables*

KKT:

$$
\begin{aligned}
Qx - A^T\lambda &= -c \\
Ax - y &= b \\
\lambda_i y_i &= 0 \ \forall\, i=1,\cdots m \\
\lambda \geq 0,&\quad y \geq 0
\end{aligned}
$$

If Q is positive semidefinite, these are sufficient conditions for optimality. This is a linear system of equations *plus*

- nonnegativity
- nonlinear complementary slackness equations!

An approach for solving the KKT conditions would be to use Phase One of the simplex method to find a feasible (basic) solution to the linear equations, with a "*restricted basis entry*" rule to enforce the complementary slackness restriction:

> "If y_i is currently in the basis at a positive level, then do not consider λ_i as a candidate for entry & vice versa. If y_i is in the basis at zero level, λ_i may enter the basis only if y_i would remain at zero level."

If Phase One with a single artificial variable is used, then at each iteration only one variable does not have a complementary variable already in the basis, so that the choice of pivot column is trivial.

If one artificial variable per row is used, then the objective function for Phase One is the sum of the artificial variables.

Minimize $\frac{1}{2}x^TQx + c^Tx$
subject to $Ax \geq b$
$x \geq 0$ ← *nonnegativity constraints included!*

If the primal variables are restricted to be nonnegative:

$$
\begin{aligned}
Qx - A^T\lambda - Iv &= -c \\
Ax - Iy &= b \\
\lambda_i y_i &= 0 \ \forall\, i=1,\ldots m \\
x_j v_j &= 0 \ \forall\, j=1,\ldots n \\
\lambda \geq 0,\ v \geq 0,\ x &\geq 0,\ y \geq 0
\end{aligned}
$$

Karush-Kuhn-Tucker conditions

EXAMPLE

$$\text{Minimize } \frac{1}{2}x^T\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}x + \begin{bmatrix} -2 \\ -6 \end{bmatrix}^T x$$

$$\text{subject to } \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} \geq \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$x_1 \geq 0,\ x_2 \geq 0$$

KKT:

$$
\begin{cases}
Ax - Iy = b \\
Qx - A^T\lambda - Iv = -c
\end{cases}
$$

x		λ		y		v		
-1	-1	0	0	-1	0	0	0	-2
1	-2	0	0	0	-1	0	0	-2
2	-2	1	-1	0	0	-1	0	2
-2	4	1	2	0	0	0	-1	6

We need a nonnegative solution satisfying the complementary slackness conditions!

x		λ		y		v		
-1	-1	0	0	-1	0	0	0	-2
1	-2	0	0	0	-1	0	0	-2
2	-2	1	-1	0	0	-1	0	2
-2	4	1	2	0	0	0	-1	6

Begin by pivoting y and v into the basis:

x		λ		y		v		
1	1	0	0	1	0	0	0	2
-1	2	0	0	0	1	0	0	2
-2	2	-1	1	0	0	1	0	-2
2	-4	-1	-2	0	0	0	1	-6

Define a (single) artificial variable (z):

x		λ		y		v		z	
1	1	0	0	1	0	0	0	0	2
-1	2	0	0	0	1	0	0	0	2
-2	2	-1	1	0	0	1	0	-1	-2
2	-4	-1	-2	0	0	0	1	-1	-6

The coefficient of z is −1 in rows having infeasibility (negative RHS), and zero otherwise.

Pivot z into the basis, by pivoting in the row having the greatest infeasibility (most negative RHS):

x		λ		y		v		z	
1	1	0	0	1	0	0	0	0	2
-1	2	0	0	0	1	0	0	0	2
-2	2	-1	1	0	0	1	0	-1	-2
2	-4	-1	-2	0	0	0	1	(-1)	-6

When we pivot in the last row, v_2 will leave the basis.

x		λ		y		v		z	
1	1	0	0	1	0	0	0	0	2
-1	2	0	0	0	1	0	0	0	2
-4	6	0	3	0	0	1	-1	0	4
-2	4	1	2	0	0	0	-1	1	6

We now have a basic nonnegative solution which satisfies complementary slackness

$x_1=x_2=0$, $v_1=4, v_2=0$
$\lambda_1=\lambda_2=0$, $y_1=2, y_2=2$
$z=6$

$\Longrightarrow$

$$\lambda_i y_i = 0 \ \forall \ i=1,2$$
$$x_j v_j = 0 \ \forall \ j=1,2$$

It would be feasible except that z>0!

In the current basis, each pair of complementary variables is represented *except* for the pair x_2 & v_2 (v_2 left the basis in the previous pivot).

Thus, of the only two candidates to enter the basis, we choose the complement of the variable which most recently left the basis, i.e., x_2 .

x		λ		y		v		z	
1	1	0	0	1	0	0	0	0	2
-1	2	0	0	0	1	0	0	0	2
-4	(6)	0	3	0	0	1	-1	0	4
-2	4	1	2	0	0	0	-1	1	6

← minimum ratio test selects this row!

v_1 leaves the basis.

x		λ		y		v		z	
(5/3)	0	0	-1/2	1	0	-1/6	1/6	0	4/3
1/3	0	0	-1	0	1	-1/3	1/3	0	2/3
-2/3	1	0	1/2	0	0	1/6	-1/6	0	2/3
2/3	0	1	0	0	0	-2/3	-1/3	1	10/3

$x_1=0$, $x_2=2/3$, $v_1=0, v_2=0$
$\lambda_1=\lambda_2=0$, $y_1=4/3, y_2=2/3$
$z=10/3$

Since v_1 just left the basis, the only candidate to enter the basis is its complement, x_1.

y_1 will leave the basis.

x		λ		y		v		z	
1	0	0	-3/10	3/5	0	-1/10	1/10	0	4/5
0	0	0	-9/10	-1/5	1	-3/10	3/10	0	2/5
0	1	0	3/10	2/5	0	1/10	-1/10	0	6/5
0	0	(1)	1/5	-2/5	0	-3/5	-2/5	1	14/5

Since y_1 just left the basis, we next enter its complement, λ_1, into the basis.

The artificial variable, z, will leave the basis, giving us a feasible solution of the KKT conditions!

The optimal solution

$x_1=4/5$, $x_2=6/5$, $y_1=0, y_2=2/5$ ← *primal*

$\lambda_1=14/5$, $\lambda_2=0$, $v_1=0, v_2=0$ ← *dual*