Successive Approximation Method

Quadratic Criterion & Linear Dynamics

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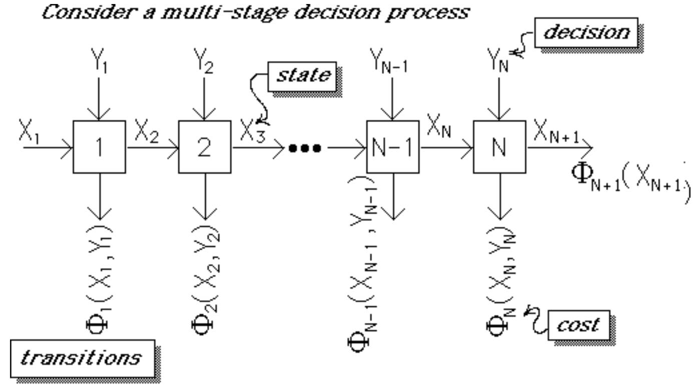
Consider a multi-stage decision process



$$X_{i+1} = \Psi_i(X_i, Y_i)$$

If the objective function Φ_i is quadratic in X & Y, and the transition function Ψ_i is linear in X & Y (the QC/LD case), we have a closed-form solution for the problem.

Otherwise, we can try successively approximating the problem by a QC/LD problem.

Step 0: "Guess" at a decision sequence $\overline{Y}_1, \overline{Y}_2, \ldots \overline{Y}_N$

Step 1: Use the transition functions together with the initial state X_1 to compute a trajectory $\overline{X}_1, \overline{X}_2, \overline{X}_3, \ldots \overline{X}_N$

Step 2: Compute $\Phi_i(\overline{X}_i, \overline{Y}_i)$, its gradient $\nabla\Phi_i(\overline{X}_i, \overline{Y}_i)$

i.e., $\frac{\partial}{\partial X_i}\Phi_i(\overline{X}_i, \overline{Y}_i), \frac{\partial}{\partial Y_i}\Phi_i(\overline{X}_i, \overline{Y}_i)$

and its Hessian matrix $\nabla^2\Phi_i(\overline{X}_i, \overline{Y}_i)$, i.e.,

$$\frac{\partial^2}{\partial X_i^2}\Phi_i(\overline{X}_i, \overline{Y}_i), \frac{\partial^2}{\partial X_i \partial Y_i}\Phi_i(\overline{X}_i, \overline{Y}_i), \ \& \frac{\partial^2}{\partial Y_i^2}\Phi_i(\overline{X}_i, \overline{Y}_i)$$

Step 3: Approximate the cost at each stage by the *Taylor series* expansion about up to & including the quadratic terms:

$$\Phi_i(X_i, Y_i) \approx \Phi_i(\overline{X}_i, \overline{Y}_i) + \frac{\partial\Phi}{\partial X_i}(\overline{X}_i, \overline{Y}_i)(X_i - \overline{X}_i) + \frac{\partial\Phi}{\partial Y_i}(\overline{X}_i, \overline{Y}_i)(Y_i - \overline{Y}_i)$$

$$+ \frac{1}{2}\left[\frac{\partial^2\Phi}{\partial X_i^2}(\overline{X}_i, \overline{Y}_i)(X_i - \overline{X}_i)^2 + \frac{\partial^2}{\partial X_i \partial Y_i}\Phi_i(\overline{X}_i, \overline{Y}_i)(X_i - \overline{X}_i)(Y_i - \overline{Y}_i) + \frac{\partial^2\Phi}{\partial Y_i^2}(\overline{X}_i, \overline{Y}_i)(Y_i - \overline{Y}_i)^2\right]$$

Step 4: Approximate the transition function by a linear function:

$$X_{i+1} = \Psi_i(X_i, Y_i) \approx \Psi_i(\overline{X}_i, \overline{Y}_i) + \frac{\partial\Psi}{\partial X_i}(\overline{X}_i, \overline{Y}_i)(X_i - \overline{X}_i) + \frac{\partial\Psi}{\partial Y_i}(\overline{X}_i, \overline{Y}_i)(Y_i - \overline{Y}_i)$$

Step 5: Use the closed-form solution to the QC/LD problem to compute the optimal decisions $\hat{Y}_1, \hat{Y}_2, \hat{Y}_3, \ldots \hat{Y}_N$

Step 6: Use the transition functions, together with initial state X_1 and decisions $\hat{Y}_1, \hat{Y}_2, \hat{Y}_3, \ldots \hat{Y}_N$ to compute the new trajectory

$$\hat{X}_{i+1} = \Psi_i(\hat{X}_i, \hat{Y}_i) \quad , i=1,2,3,\ldots N$$

Step 7: If the termination criterion is not satisfied, let $\overline{X} = \hat{X}$ and $\overline{Y} = \hat{Y}$, and return to step 2.

Example

Minimize $Y_1^4 + X_2^4 + Y_2^4 + X_3^4$

subject to

$$\begin{cases} X_{i+1} = X_i^2 + 4Y_i, \ i=1,2 \\ X_1 = 2 \end{cases}$$

$$\overline{Y}_1 = -1, \ \overline{Y}_2 = -0.1$$

Problem Statement

The example problem in the SAMDP.FNS file is:

```
Minimize  (Y[1]*4) + (X[2]*4) + (Y[2]*4) + (X[3]*4)
        subject to
                  X[T+1] ← (X[T]*2) + (4×Y[T]),  T=1,2
                  X[1] = 2
```

A good starting 'guess' at the optimal decisions is

```
Y = ¯1, ¯0.1
```


Test Problem □ Sequential Approximation Method

Number of stages: N = 2

Objective Function

```
Z←T FN XY
⍝
⍝       Objective Function for SAMDP
⍝
Z←(0 1 1)[T]×XY[1]*4
→0 IF N<T
Z←Z+(1 1 0)[T]×XY[2]*4
```


Gradient of Objective Function

```
G←T GRADF XY
⍝
⍝     Gradient of objective function for example problem
⍝
G←(0 1 1)[T]×4×XY[1]*3
→Last IF N<T   ⋄ G←G,0  ⋄ →0
Last:G←G,(1 1 0)[T]×4×XY[2]*3
```


Hessian of Objective Function

```
H←T HESSIAN XY
⍝
⍝     Hessian matrix for objective function
⍝            of example problem
⍝
H←2 2⍴0
H[1;1]←(0 1 1)[T]×12×XY[1]*2
→0 IF N<T
H[2;2]←(1 1 0)[T]×12×XY[2]*2
```


Transition function

```
Z←T PHI XY
⍝
⍝       Transition function for example problem
⍝
Z←(XY[1]*2) + 4×XY[2]
```

Gradient of Transition Function

```
G←T GRADPHI XY
⍝
⍝     Gradient of transition function for example problem
⍝
G←(2×XY[1]),4
```


Iteration 1

QC/LD Approximation at:

i	1	2	3
X[i]	2	0	¯0.4
Y[i]	¯1	¯0.1	0

Objective function value is 1.0257

T	A	B	C	D	E	F	G	H	K
0	0	0	6	0	12	7	4	4	¯4
1	0	0	0.06	0	0.012	0.0007	0	4	0
2	0.96	0	0	0.512	0	0.0768	0	0	0

Solution of QC/LD Approximation is:

i	1	2	3
X[i]	2	0	¯0.267185
Y[i]	¯1	¯0.0667964	0

Sum of |Y-Y'| = 0.0332036

Iteration 2

QC/LD Approximation at:

i	1	2	3
X[i]	2	0	¯0.267185
Y[i]	¯1	¯0.0667964	0

Objective function value is 1.00512

A	B	C	D	E	F	G	H	K
0	0	6	0	12	7	4	4	¯4
0	0	0.02677	0	0.003576	0.0001393	0	4	0
0.428328	0	0	0.152591	0	0.0152888	0	0	0

Solution of QC/LD Approximation is:

i	1	2	3
X[i]	2	0	¯0.17847
Y[i]	¯1	¯0.0446175	0

Sum of |Y-Y'| = 0.0221788

Iteration 3

QC/LD Approximation at:

```
X[i] 2   0           ¯0.17847
Y[i]¯1  ¯0.0446175   0
```

Objective function value is 1.00102

T	A	B	C	D	E	F	G	H	K
0	0E0	0	6E0	0E0	1.2E1	7E0	4	4	¯4
1	0E0	0	1.19444E¯2	0E0	1.06586E¯3	2.77409E¯5	0	4	0
2	1.9111E¯1	0	0E0	4.54765E¯2	0E0	3.04358E¯3	0	0	0

Solution of QC/LD Approximation is:

```
i    1  2           3
X[i] 2  0          ¯0.119212
Y[i]¯1 ¯0.0298029   0
```

Sum of |Y-Y'| = 0.0148146

Iteration 4

QC/LD Approximation at:

```
X[i] 2   0           ¯0.119212
Y[i]¯1  ¯0.0298029   0
```

Objective function value is 1.0002

T	A	B	C	D	E	F	G	H	K
0	0E0	0	6E0	0E0	1.2E1	7E0	4	4	¯4
1	0E0	0	5.32928E¯3	0E0	3.17656E¯4	5.52246E¯6	0	4	0
2	8.52684E¯2	0	0E0	1.35533E¯2	0E0	6.05892E¯4	0	0	0

Solution of QC/LD Approximation is:

```
i    1  2           3
X[i] 2  0          ¯0.079629
Y[i]¯1 ¯0.0199073   0
```

Sum of |Y-Y'| = 0.00989565