Quadratic Criterion & Linear Dynamics

Multi-Dimensional Case

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State Variable X_t is n-dimensional
Decision Variable Y_t is m-dimensional
given initial state X_1,

$$\text{Minimize} \sum_{t=1}^{T} \left\{ X_t' A_t X_t + X_t' B_t Y_t + Y_t' C_t Y_t + d_t X_t + e_t Y_t + f_t \right\} + X_{T+1}' A_{T+1} X_{T+1} + d_{T+1} X_{T+1} + f_{T+1}$$

where $X_{t+1} = G_t X_t + H_t Y_t + k_t$, $t=1,2,\ldots T$

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given initial state X^1,

$$\text{Minimize} \sum_{t=1}^{T} \left\{ X_t' A_t X_t + X_t' B_t Y_t + Y_t' C_t Y_t + d_t X_t + e_t Y_t + f_t \right\} + X_{T+1}' A_{T+1} X_{T+1} + d_{T+1} X_{T+1} + f_{T+1}$$

where $X_{t+1} = G_t X_t + H_t Y_t + k_t$, $t=1,2,\ldots T$

A_t is nxn, B_t is nxm, C_t is mxm, d_t is n-vector, e_t is m-vector, etc.

DP approach

Optimal Value Function:

$V_t(X)$ = minimum cost of stages t, t+1, ... T+1 given that the state at stage t is X

Can be evaluated recursively, and will be of the form

$$V_t(X) = X' P_t X + q_t X + r_t$$

(Clearly true for t=T+1)

To compute recursively the quantities P, q, and r, at stage t, first compute

$$\begin{cases} S_t = C_t + H_t' P_{t+1} H_t & (m\times m) \\ t_t = e_t + 2H_t' P_{t+1} k_t + H_t' q_{t+1} & (m\text{-vector}) \\ U_t = B_t' + 2H_t' P_{t+1} G_t & (m\times n) \end{cases}$$

where $P_{T+1} = A_{T+1}$, $q_{T+1} = d_{T+1}$, $r_{T+1} = f_{T+1}$

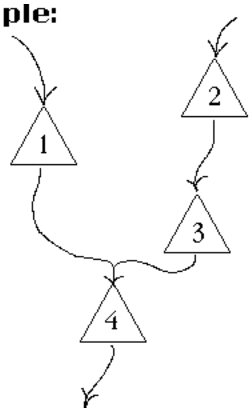
Then compute

$$\begin{cases} P_t = A_t + G_t' P_{t+1} G_t - \frac{1}{4} U_t' S_t^{-1} U_t \\ q_t = d_t + 2k_t' P_{t+1} G_t + q_{t+1} G_t - \frac{1}{2} t_t' S_t^{-1} U_t \\ r_t = f_t + k_t' P_{t+1} k_t + q_{t+1} k_t + r_{t+1} - \frac{1}{4} t_t' S_t^{-1} t_t \end{cases}$$

Optimal decisions:

$$Y_t = -\frac{1}{2} S_t^{-1} \left[U_t X_t + t_t \right]$$

Example:



Multi-Reservoir Operation

Example: Multi-Reservoir Operation

State Variable

X_{ti} = volume of water in reservoir #i at the start of period t

Decision Variable

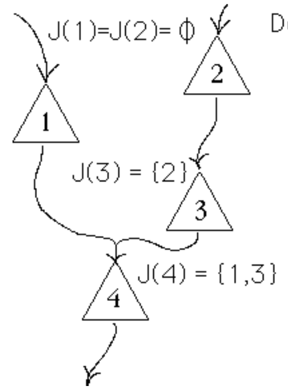
Y_{ti} = volume released from reservoir #i during period t

Let $\hat{X}_{ti}$ = target storage volume

$\hat{Y}_{ti}$ = target release

Our objective is to minimize the weighted sum of the squared deviations from the targets:

$$\text{Minimize} \sum_i \sum_t \alpha_{it}\left(\hat{Y}_{it} - Y_{it}\right)^2 + \beta_{it}\left(\hat{X}_{it} - X_{it}\right)^2$$



Define

$J(i)$ = set of reservoirs immediately upstream from reservoir #i

e_{it} = evaporation rate of reservoir #i in period t

I_{it} = inflow (exclusive of upstream releases)

Transition Equations

$$X_{i,t+1} = (1 - e_{it})\, X_{it} + \sum_{j\in J(i)} Y_{jt} - Y_{it} + I_{it}$$

linear dynamics!

In reality, evaporation losses are nonlinear functions of volume, being approximately linear in the surface area of the reservoir.

QC/LD problem

$$\text{Minimize} \sum_i \sum_t \alpha_{it}\left(\hat{Y}_{it} - Y_{it}\right)^2 + \beta_{it}\left(\hat{X}_{it} - X_{it}\right)^2$$

subject to

$$X_{i,t+1} = (1 - e_{it})\, X_{it} + \sum_{j\in J(i)} Y_{jt} - Y_{it} + I_{it}$$

where the initial storage volumes X_{i1} *are given.*

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}_{t+1} = \begin{bmatrix} 0.95 & 0 & 0 & 0 \\ 0 & 0.98 & 0 & 0 \\ 0 & 0 & 0.97 & 0 \\ 0 & 0 & 0 & 0.92 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}_t + \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ +1 & 0 & +1 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}_t + \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}_t$$

Transition Equations

INFLOWS

100	80	30	40
90	75	25	35
110	90	30	50

(expected) inflows into 4 reservoirs during the 3 periods

EVAP

0.05	0.02	0.03	0.08

Evaporation rates of the three reservoirs

RESERVOIR∆ADJACENCY

0	0	0	1
0	0	1	0
0	0	0	1
0	0	0	0

Adjacency Matrix

STORAGE∆TARGET

90	75	100	200
100	80	110	190
80	70	90	195
90	75	100	180

RELEASE∆TARGET

100	75	100	250
90	70	90	220
110	80	110	250

PEN∆S

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

α

PEN∆R

1	1	1	1
1	1	1	1
1	1	1	1

β

INITIAL∆STORAGE

100	80	90	220

a = Quadratic Cost Coefficients of X

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

b = Cost Coefficients of X×Y

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

c = Quadratic Cost Coefficients of Y

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

d = Linear Cost Coefficients of X

¯180	¯150	¯200	¯400
¯200	¯160	¯220	¯380
¯160	¯140	¯180	¯390
¯180	¯150	¯200	¯360

e = Linear Cost Coefficients of Y

¯200	¯150	¯200	¯500
¯180	¯140	¯180	¯440
¯220	¯160	¯220	¯500

f = Constants in Cost Function

151850 134100 150525 56125

g = Linear Coefficients of X in Transition Eqn.

0.95	0	0	0
0	0.98	0	0
0	0	0.97	0
0	0	0	0.92
0.95	0	0	0
0	0.98	0	0
0	0	0.97	0
0	0	0	0.92
0.95	0	0	0
0	0.98	0	0
0	0	0.97	0
0	0	0	0.92

h = Linear Coefficients of Y in Transition Eqn.

¯1	0	0	0
0	¯1	0	0
0	1	¯1	0
1	0	1	¯1
¯1	0	0	0
0	¯1	0	0
0	1	¯1	0
1	0	1	¯1
¯1	0	0	0
0	¯1	0	0
0	1	¯1	0
1	0	1	¯1

k = Constants in Transition Eqn.

100	80	30	40
90	75	25	35
110	90	30	50

Initialize:

$$P_{T+1} = A_{T+1},\ q_{T+1} = d_{T+1},\ r_{T+1} = f_{T+1}$$

For t=T, T-1, ...2,1 , *compute:*

$$S_t = C_t + H_t' P_{t+1} H_t$$

$$t_t = e_t + 2H_t' P_{t+1} k_t + H_t' q_{t+1}$$

$$U_t = B_t' + 2 H_t' P_{t+1} G_t$$

$$P_t = A_t + G_t' P_{t+1} G_t - \frac{1}{4} U_t' S_t^{-1} U_t$$

$$q_t = d_t + 2k_t' P_{t+1} G_t + q_{t+1} G_t - \frac{1}{2} t_t' S_t^{-1} U_t$$

$$r_t = f_t + k_t' P_{t+1} k_t + q_{t+1} k_t + r_{t+1} - \frac{1}{4} t_t' S_t^{-1} t_t$$

Results

Array p

1.75247	0.150234	0.18517	0.26271
0.150234	1.97463	0.489517	0.183195
0.18517	0.489517	1.65924	0.234675
0.26271	0.183195	0.234675	1.36283
1.69534	0.0988062	0.137776	0.22675
0.0988062	1.85358	0.396428	0.135955
0.137776	0.396428	1.58418	0.193948
0.22675	0.135955	0.193948	1.33659
1.52403	0.0300323	0.0594516	0.140968
0.0300323	1.58863	0.214652	0.0581677
0.0594516	0.214652	1.42492	0.115148
0.140968	0.0581677	0.115148	1.27303
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

Array q

¯506.702	¯493.292	¯541.144	¯685.525
¯457.738	¯419.76	¯485.065	¯617.351
¯324.565	¯288.265	¯347.403	¯543.135
¯180	¯150	¯200	¯360

Array r

141203 122073 92082.3 56125

Optimal total cost

$$V_1(X) = X' P_1 X + q_1 X + r_1$$

$$= \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \end{bmatrix} \begin{bmatrix} 1.75247 & 0.150234 & 0.18517 & 0.26271 \\ 0.150234 & 1.97463 & 0.489517 & 0.183195 \\ 0.18517 & 0.489517 & 1.65924 & 0.234675 \\ 0.26271 & 0.183195 & 0.234675 & 1.36283 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

$$+ \begin{bmatrix} -506.702 & -493.292 & -541.144 & -685.525 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + 141203$$

$$= 1.75247X_1^2 + 2(0.150234)X_1 X_2 + \ldots + 1.36283X_4^2$$

$$-506.702X_1 - \ldots - 685.525X_4 + 141203$$

where INITIAL∆STORAGE

X = 100 80 90 220

Array s

3.57843	0.0190237	1.05367	¯1.10984
0.0190237	3.6449	¯1.12976	¯0.0579936
1.05367	¯1.12976	3.53287	¯1.14264
¯1.10984	¯0.0579936	¯1.14264	2.33659
3.51513	0.0275613	1.07637	¯1.13206
0.0275613	3.58425	¯1.15329	¯0.0569806
1.07637	¯1.15329	3.46766	¯1.15788
¯1.13206	¯0.0569806	¯1.15788	2.27303
3	0	1	¯1
0	3	¯1	0
1	¯1	3	¯1
¯1	0	¯1	2

Array t

¯555.229	¯364.751	¯348.169	¯68.3163
¯561.273	¯335.438	¯368.969	¯25.8335
¯520	¯330	¯340	¯240

Array u

¯2.79031	0.0728113	0.108974	2.04211
0.0740428	¯2.85602	2.30423	0.106708
0.169051	¯0.510528	¯2.69705	2.10247
¯0.430826	¯0.266471	¯0.37626	¯2.45933
¯2.62782	0.0551455	0.108052	2.083
0.0558968	¯2.693	2.34793	0.104844
0.154881	¯0.306708	¯2.54096	2.13051
¯0.267839	¯0.114009	¯0.223388	¯2.34238
¯1.9	0	0	1.84
0	¯1.96	1.94	0
0	0	¯1.94	1.84
0	0	0	¯1.84

$$Y_t = -\frac{1}{2} S_t^{-1} \left[U_t X_t + t_t\right]$$

Optimal Decisions

$$= \frac{-1}{2}\begin{bmatrix} 3.578 & 0.019 & 1.053 & -1.109 \\ 0.019 & 3.644 & -1.129 & -0.057 \\ 1.053 & -1.129 & 3.532 & -1.142 \\ -1.109 & -0.057 & -1.142 & 2.336 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -2.7903 \\ 0.0728 \\ 0.1089 \\ 2.0421 \end{bmatrix}' \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} -555.229 \\ -364.751 \\ -348.169 \\ -68.316 \end{bmatrix} \right\}$$

Forward Computations of Optimal States & Decisions

Stage 1

State X= 100 80 90 220
Decision Y= 99.2614 82.817 98.9775 249.029

Stage 2

State X= 95.7386 75.583 101.14 191.61
Decision Y= 95.4911 73.3701 99.8843 217.195

Stage 3

State X= 85.4606 75.7012 96.5911 189.461

Optimal value is 1200.51

Optimal Solution QC/LD Problem

State Variables (Storage Volumes)

t \ Reservoir	1	2	3	4
1	100	80	90	220
2	95.7386	75.583	101.14	191.61
3	85.4606	75.7012	96.5911	189.461

Decision Variables

t \ Reservoir	1	2	3	4
1	99.2614	82.817	98.9775	249.029
2	95.4911	73.3701	99.8843	217.195