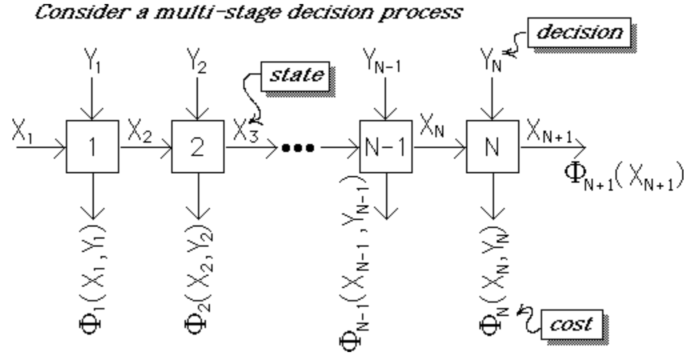
Quadratic Criterion & Linear Dynamics

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

1-dimensional case

author

Consider a multi-stage decision process



Let X_i = state of system at stage i
Y_i = decision at stage i

continuous variables

General QC/LD problem

$$\text{Minimize} \sum_{i=1}^{N}\left\{A_iX_i^2 + B_iX_iY_i + C_iY_i^2 + D_iX_i + E_iY_i + F_i\right\} + A_{N+1}X_{N+1}^2 + D_{N+1}X_{N+1} + F_{N+1}$$

subject to

$$X_{i+1} = G_iX_i + H_iY_i + K_i\,, \qquad i=2,3,\cdots N$$

- Derivation of Solution for Simpler Version
- Closed-form Solution of General Problem
- Example APL Output
- Certainty Equivalence

Let's begin with a simpler version of the problem:

$$\text{Minimize} \sum_{i=1}^{N}\{A_iX_i^2 + C_iY_i^2\} + A_{N+1}X_{N+1}^2$$

Quadratic criterion

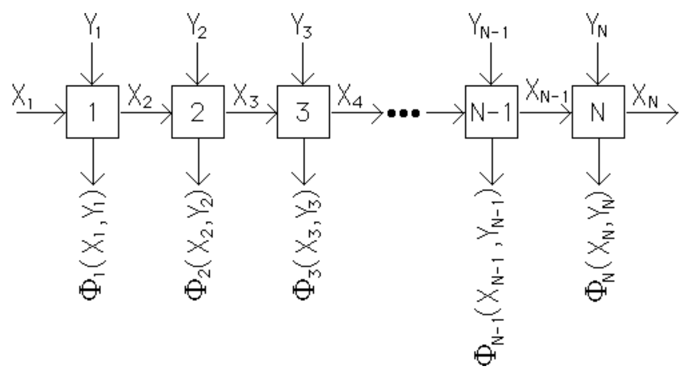
where

$$X_{i+1} = G_iX_i + H_iY_i, \qquad i=1,\ldots N$$

Linear Dynamics

Assume $A_i \geq 0$ & $C_i \geq 0$

Convexity



DP solution

Optimal value function

$V_i(X)$ = minimum cost of the remaining process if it starts stage i in state X

$$= \underset{y}{\text{minimum}}\{A_iX^2 + C_iY^2 + V_{i+1}(G_iX + H_iY)\}$$

i=1,2,...N

$$V_{N+1}(X) = A_{N+1}X^2$$

The problem at the last stage:

$$V_N(X) = \min_y\{A_NX^2 + C_NY^2 + V_{N+1}(G_NX + H_NY)\}$$
$$= \min_y\{A_NX^2 + C_NY^2 + A_{N+1}(G_NX + H_NY)^2\}$$
$$= \min_y\{A_NX^2 + C_NY^2 + A_{N+1}G_N^2X^2 + 2A_{N+1}G_NH_NXY + A_{N+1}H_N^2Y^2\}$$

$$V_N(X) = \min_Y \{A_N X^2 + C_N Y^2 + A_{N+1} G_N^2 X^2 + 2A_{N+1} G_N H_N XY + A_{N+1}^2 H_N^2 Y^2\}$$

Set the partial derivative of the minimand equal to zero:

$$2C_N Y + 2A_{N+1} G_N H_N X + 2A_{N+1} H_N^2 Y = 0$$

$$\Rightarrow \boxed{Y = -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2}}$$

$$\boxed{Y = -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2}}$$

is a minimizer if the second derivative is positive, i.e., if

$$C_N + A_{N+1} H_N^2 > 0$$

which we will assume.

Substituting $\boxed{Y = -\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2}}$

into $A_N X^2 + C_N Y^2 + A_{N+1} G_N^2 X^2 + 2A_{N+1} G_N H_N XY + A_{N+1} H_N^2 Y^2$

yields

$$V_N(X) = A_N X^2 + C_N\left(-\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2}\right)^2 + A_{N+1} G_N^2 X^2 + 2A_{N+1} G_N H_N X\left(-\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2}\right) + A_{N+1} H_N^2\left(-\frac{A_{N+1} G_N H_N X}{C_N + A_{N+1} H_N^2}\right)^2$$

$$V_N(X) = \underbrace{\left\{A_N + A_{N+1} G_N^2 - \frac{A_{N+1}^2 G_N^2 H_N^2}{C_N + A_{N+1} H_N^2}\right\}}_{} X^2$$

$$V_N(X) = P_N X^2$$

Note that V_N *is a quadratic function of X, as was* $V_{N+1}(X) = A_{N+1} X^2$

We can now use $V_N(X)$ *to find* $V_{N-1}(X)$...
The same formulae will result, except that P_N *replaces* A_{N+1}, A_{N-1} *replaces* A_N, *etc.*

$$V_{N-1}(X) = P_{N-1} X^2$$

where

$$P_{N-1} = A_{N-1} + P_N G_{N-1}^2 - \frac{P_N^2 G_{N-1}^2 H_{N-1}^2}{C_{N-1} + P_N H_{N-1}^2}$$

optimal decision: $\boxed{Y = -\frac{P_N G_{N-1} H_{N-1}}{C_{N-1} + P_N H_{N-1}^2} X}$

Clearly, we can repeat this procedure to get
$V_{N-2}(X)$, $V_{N-3}(X)$, ... $V_2(X)$, $V_1(X)$
where in general,

$$V_i(X) = P_i X^2$$

where

$$P_i = A_i + P_{i+1} G_i^2 - \frac{P_{i+1}^2 G_i^2 H_i^2}{C_i + P_{i+1} H_i^2}$$

$$Y = -\frac{P_{i+1} G_i H_i}{C_i + P_{i+1} H_i^2} X$$

Example Given initial state X_1, select Y_1, Y_2, and Y_3 to

$$\text{Minimize } Y_1^2 + 12X_2^2 + 2Y_2^2 + 2X_3^2 + Y_3^2 + \frac{1}{4} X_4^2$$

where

$$\begin{cases} X_2 = \frac{1}{2} X_1 + \frac{1}{6} Y_1 \\ X_3 = 3X_2 + \frac{1}{2} Y_2 \\ X_4 = 4X_3 + 2Y_3 \end{cases}$$

Minimize $Y_1^2 + 12X_2^2 + 2Y_2^2 + 2X_3^2 + Y_3^2 + \frac{1}{4}X_4^2$

where
$$\begin{cases} X_2 = \frac{1}{2}X_1 + \frac{1}{6}Y_1 \\ X_3 = 3X_2 + \frac{1}{2}Y_2 \\ X_4 = 4X_3 + 2Y_3 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = 0, & C_1 = 1, & G_1 = 1/2, & H_1 = 1/6 \\ A_2 = 12, & C_2 = 2, & G_2 = 3, & H_2 = 1/2 \\ A_3 = 2, & C_3 = 1, & G_3 = 4, & H_3 = 2 \\ A_4 = \frac{1}{4} \end{cases}$$

Backward Computation

$V_i(X) = P_iX^2$

where
$$P_i = A_i + P_{i+1}G_i^2 - \frac{P_{i+1}^2G_i^2H_i^2}{C_i + P_{i+1}H_i^2}$$
$$Y = -\frac{P_{i+1}G_iH_i}{C_i + P_{i+1}H_i^2}X$$

$$P_3 = 2 + (1/4)\,4^2 - \frac{(1/4)^2 4^2 2^2}{1 + (1/4)2^2} = 4, \quad \text{where } P_4 = A_4 = \frac{1}{4}$$
$$Y_3 = -\frac{(1/4)(4)(2)}{1 + (1/4)2^2}X_3 = -X_3$$

Backward Computation

$V_i(X) = P_iX^2$

where
$$P_i = A_i + P_{i+1}G_i^2 - \frac{P_{i+1}^2G_i^2H_i^2}{C_i + P_{i+1}H_i^2}$$
$$Y = -\frac{P_{i+1}G_iH_i}{C_i + P_{i+1}H_i^2}X$$

$$P_2 = 12 + 4\left(4^2\right) - \frac{4^2 3^2 (1/2)^2}{2 + 4\,(1/2)^2} = 36,$$
$$Y_2 = -\frac{(4)(3)(1/2)}{2 + 4\,(1/2)^2}X_2 = -2X_2$$

Backward Computation

$V_i(X) = P_iX^2$

where
$$P_i = A_i + P_{i+1}G_i^2 - \frac{P_{i+1}^2G_i^2H_i^2}{C_i + P_{i+1}H_i^2}$$
$$Y = -\frac{P_{i+1}G_iH_i}{C_i + P_{i+1}H_i^2}X$$

$$P_1 = 0 + 36(1/2)^2 - \frac{36^2(1/2)^2(1/6)^2}{1 + 36\,(1/6)^2} = 9/2,$$
$$Y_1 = -\frac{(36)(1/2)(1/6)}{1 + 36\,(1/6)^2}X_1 = -\,3/2\,X_1$$

Optimal value: $V_1(X) = 9/2\,X^2 \Rightarrow V_1(2) = 9/2\,2^2 = 18$

Now perform a "forward computation":

Given $X_1 = 2$, $\Rightarrow Y_1 = -\,3/2\,X_1 = -3$ $\Rightarrow$ $X_2 = 1/2\,X_1 + 1/6\,Y_1 = 1/2$ $\Rightarrow$ $Y_2 = -2X_2 = -1$

$\Rightarrow$ $X_3 = 3X_2 + 1/2\,Y_2 = 1$ $\Rightarrow$ $Y_3 = -X_3 = -1$ $\Rightarrow$ $X_4 = 4X_3 + 2Y_3 = 2$

General QC/LD problem

Minimize $\sum_{i=1}^{N}\left\{A_iX_i^2 + B_iX_iY_i + C_iY_i^2 + D_iX_i + E_iY_i + F_i\right\} + A_{N+1}X_{N+1}^2 + D_{N+1}X_{N+1} + F_{N+1}$

subject to

$$X_{i+1} = G_iX_i + H_iY_i + K_i, \quad i=2,3,\ldots N$$

Using the same method as before, we can derive closed-form expressions for the optimal value and decisions.

Optimal Value Function $V_i(X) = P_iX_i^2 + Q_iX_i + R_i$

where
$$P_i = A_i + P_{i+1}G_i^2 - \frac{[B_i + 2P_{i+1}G_iH_i]^2}{4\,[C_i + P_{i+1}H_i^2]}$$
$$Q_i = D_i + 2P_{i+1}K_iG_i + Q_{i+1}G_i - \frac{(B_i + 2P_{i+1}G_iH_i)(E_i + 2P_{i+1}H_iK_i + Q_{i+1}H_i)}{2(C_i + P_{i+1}H_i^2)}$$
$$R_i = F_i + P_{i+1}K_i^2 + Q_{i+1}K_i + R_{i+1} - \frac{(E_i + 2P_{i+1}H_iK_i + Q_{i+1}H_i)^2}{4\,[C_i + P_{i+1}H_i^2]}$$

$P_N = A_{N+1}$
$Q_N = D_{N+1}$
$R_N = F_{N+1}$

Optimal Decisions

$$Y_i = -\frac{(B_i + 2P_{i+1}G_iH_i)X + E_i + 2P_{i+1}H_iK_i + Q_{i+1}H_i}{2(C_i + P_{i+1}H_i^2)}$$

APL output

Cost Data

i	A	B	C	D	E	F
0	0	0	1	0	0	0
1	12	0	2	0	0	0
2	2	0	1	0	0	0

```
where
  A[i] = coefficient of X[i]*2      D[i] = coefficient of X[i]
  B[i] = coefficient of X[i]×Y[i]   E[i] = coefficient of Y[i]
  C[i] = coefficient of Y[i]*2      F[i] = constant
Cost of final stage: 0.25×X[N]*2 + 0×X[N] + 0
```


Transition data

i	G	H	K
0	0.5	0.1666666667	0
1	3	0.5	0
2	4	2	0

```
where

  X[i+1] = (G[i]×X[i]) + (H[i]×Y[i]) + K[i]
```


i	P	Q	R	S	T
0	4.5	0	0	¯1.5	0
1	36	0	0	¯2	0
2	4	0	0	¯1	0
3	0.25	0	0	0	0

```
Optimal decision  Y[i] = (S[i]×X[i]) + T[i]

Optimal value     V[i] = (P[i]×X[i]*2) + (Q[i]×X[i]) + R[i]
```


i	Xi	Yi
0	2	¯3
1	0.5	¯1
2	1	¯1
3	2	

```
     X[i] = state variable,
and
     Y[i] = decision variable,
at stage i

     Optimal Cost: 18
```


Certainty Equivalence

Consider the following stochastic version of the QC/LD problem, with a random additive term Z_i in the linear dynamics (transition equation)

$$\text{Minimize} \sum_{i=1}^{N} \left\{ A_i X_i^2 + B_i X_i Y_i + C_i Y_i^2 + D_i X_i + E_i Y_i + F_i \right\} + A_{N+1} X_{N+1}^2 + D_{N+1} X_{N+1} + F_{N+1}$$

$$\text{subject to} \quad X_{i+1} = G_i X_i + H_i Y_i + Z_i, \quad i=2,3,\ldots N$$

Transition Equations

$$X_{i+1} = G_i X_i + H_i Y_i + Z_i$$

where Z_i , i=1, 2, ... N are independent random variables, with

$$E(Z_i) = \mu_i$$

$$Var(Z_i) = \sigma_i^2$$

We assume that Y_i must be selected *before* the random variable Z_i is observed.

Define an optimal value function:

$W_i(X)$ = minimum *expected* cost of the remaining process, if we start stage i in state X, and have not yet learned the value of Z_i

Solution

$$W_i(X) = P_i X_i^2 + Q_i X_i + R_i$$

where

$$P_i = A_i + P_{i+1} G_i^2 - \frac{[B_i + 2P_{i+1} G_i H_i]^2}{4[C_i + P_{i+1} H_i^2]}$$

$$Q_i = D_i + 2P_{i+1}\mu_i G_i + Q_{i+1} G_i - \frac{[B_i + 2P_{i+1} G_i H_i][E_i + 2P_{i+1} H_i \mu_i + Q_{i+1} H_i]}{2[C_i + P_{i+1} H_i^2]}$$

$$R_i = F_i + P_{i+1}[\mu_i^2 + \sigma_i^2] + Q_{i+1}\mu_i + R_{i+1} - \frac{(E_i + 2P_{i+1} H_i \mu_i + Q_{i+1} H_i)^2}{4[C_i + P_{i+1} H_i^2]}$$

Optimal Decision

$$Y_i = -\frac{2P_{i+1} G_i H_i X + 2P_{i+1} H_i \mu_i + Q_{i+1} H_i}{2[C_i + P_{i+1} H_i^2]}$$

The closed-form solution to this stochastic problem is identical to the deterministic version of the problem with $K_i = E[Z_i] = \mu_i$ except that the equation for R_i differs by a term $P_{i+1}\sigma_i^2$

The value of R_i does not enter into the computation of the optimal decision Y_i, however.

Certainty Equivalence

The optimal policy for the stochastic problem is the same as that of the deterministic problem, with the random variable replaced by its expected value.

(The cost of the optimal policy is increased, due to the different formula for R_i, reflecting the cost due to randomness.)