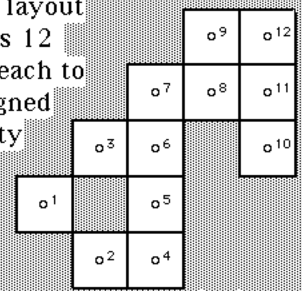
Quadratic Assignment Problem: a Simulated Annealing Algorithm

author

This Hypercard stack was prepared by:
Dennis Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

Example:
A floor layout contains 12 rooms, each to be assigned a facility



Define a binary decision variable for each combination of facility and location:

$$X_{ia} = \begin{cases} 1 & \text{if facility i is located at "a"} \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum_{i=1}^{n} X_{ia} = 1$ for each location a=1, … n

(each location is to be assigned exactly one facility)

and $\sum_{a=1}^{n} X_{ia} = 1$ for each facility i=1, … n

(each facility is to be assigned to a location)

Distance Matrix

D (from → to)

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	2	2	3	2	3	4	5	6	5	6	7
2	2	0	2	1	2	3	4	5	6	5	6	7
3	2	2	0	3	2	1	2	3	4	3	4	5
4	3	1	3	0	1	2	3	4	5	4	5	6
5	2	2	2	1	0	1	2	3	4	3	4	5
6	3	3	1	2	1	0	1	2	3	2	3	4
7	4	4	2	3	2	1	0	1	2	3	2	3
8	5	5	3	4	3	2	1	0	1	2	1	2
9	6	6	4	5	4	3	2	1	0	3	2	1
10	5	5	3	4	3	2	3	2	3	0	1	2
11	6	6	4	5	4	3	2	1	2	1	0	1
12	7	7	5	6	5	4	3	2	1	2	1	0

(These are the "rectangular" distances between centers of the areas)

Interfacility Flow Matrix

F (from → to)

	A	B	C	D	E	F	G	H	I	J	K	L
A			3						4		2	
B									4			
C	3					5	1			1	1	4
D									1			
E								1				3
F			5							3		3
G			1								3	
H					1							
I	4	4		1						4		
J			1			3			4		1	
K	2		1				3			1		
L			4		3	3						

Density = 25.76%

If facility i is located at location "a", and facility j at "b", then the cost of the flow between this pair of facilities is assumed to be:

$$F_{ij} D_{ab}$$

Minimize

The optimization problem is to

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} F_{ij} D_{ab} X_{ia} X_{jb}$$

Note that the cost function is not linear, but QUADRATIC!

subject to

$\sum_{i=1}^{n} X_{ia} = 1$ for each location a=1, … n

(each location is to be assigned exactly one facility)

$\sum_{a=1}^{n} X_{ia} = 1$ for each facility i=1, … n

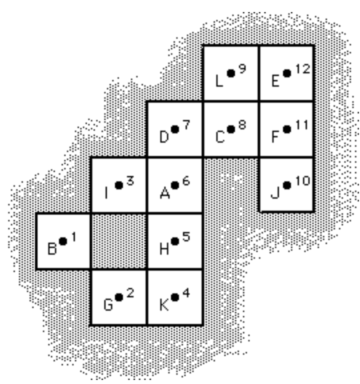
(each facility is to be assigned to a location)

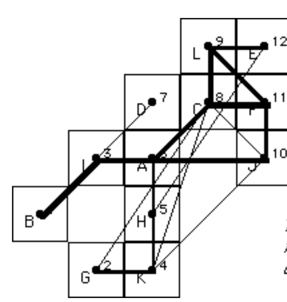
$X_{ia} \in \{1,0\}$ for each i=1, … n & a=1, …n

A heuristic solution:

Facility	Location
A	6
B	1
C	8
D	7
E	12
F	11
G	2
H	5
I	3
J	10
K	4
L	9

Cost: 160 sum of weighted distances





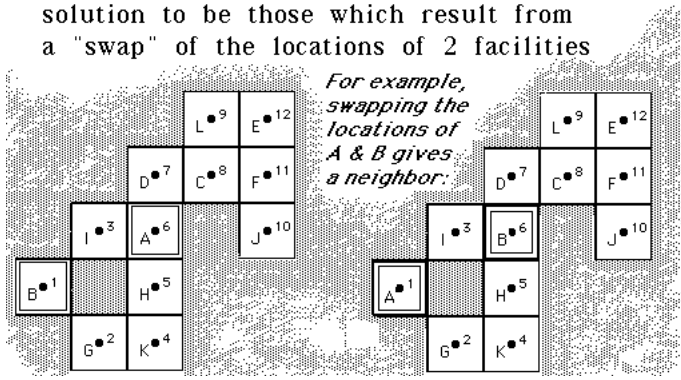
Material Flows

The thickness of the line indicates the magnitude of the flow

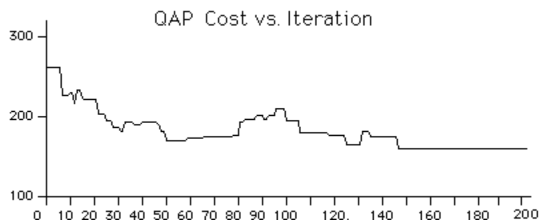
"Simulated Annealing"

- a heuristic search approach
- a move is made to any neighboring solution with equal or lower cost
- if the neighbor increases the cost by $\Delta > 0$, then the move is accepted with probability $P\{\text{accept } \Delta\} = e^{-\Delta/T}$ where T is the current "temperature" of the system
- the system is "cooled" according to some "cooling schedule"

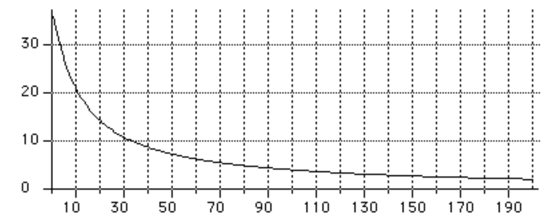
We will consider the neighbors of a solution to be those which result from a "swap" of the locations of 2 facilities



A typical simulated annealing result:



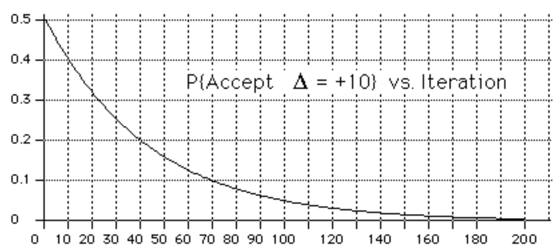
After each iteration, the temperature is reduced, according to a "cooling schedule"



$$T_{i+1} = \frac{T_i}{1+\beta T_i} \quad \text{where} \quad \beta = \frac{(T_0 - T_f)}{M\, T_0 T_f}$$

& T_0 = initial temperature, T_f = final temperature, M = # of iterations

As the system "cools", the probability of accepting an increase (of 10) decreases:



$$P\{\text{accept } \Delta\} = e^{-\Delta/T}$$

The first 15 iterations of a simulated annealing:

Iteration #	Temp	Z	Swap pair	Δ	P{accept}	Accept ?	
1	14.42695	262	(1↔ 3)	44	0.0474		
2	13.96311	262	(1↔ 9)	12	0.4234		
3	13.52816	262	(1↔11)	0	1.0000	Y	
4	13.11949	262	(2↔ 9)	18	0.2536		
5	12.73479	262	(3↔ 6)	28	0.1109		
6	12.37200	262	(3↔ 7)	¯34	1.0000	Y	↓
7	12.02932	228	(3↔10)	12	0.3688		
8	11.70510	228	(3↔11)	84	0.0008		
9	11.39791	228	(3↔12)	2	0.8391	Y	↑
10	11.10642	230	(4↔ 9)	0	1.0000	Y	
11	10.82948	230	(5↔ 8)	¯12	1.0000	Y	↓
12	10.56600	218	(5↔12)	16	0.2200	Y	↑
13	10.31505	234	(6↔10)	20	0.1439		
14	10.07574	234	(6↔12)	¯10	1.0000	Y	↓
15	9.84728	224	(7↔11)	¯2	1.0000	Y	↓

← *a swap which results in an increase is accepted!* (iterations 9 and 12)

QAP

Author Connolly, David T.

Title An improved annealing scheme for the QAP

Pub. European Journal of Operational Research, Volume 46 (1990), pp. 93-100

Notes

Key Simulated annealing, heuristics