Portfolio Optimization via Quadratic Programming

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

author

A fixed sum of money F is to be allocated among n investments, each of which has a known history of returns during the previous p periods

r_{ik} = return per dollar invested in investment #i during period k, i=1,...n; k=1,...p

x_i = amount of money to be allocated to investment #i

EXAMPLE n=3, p=5

Investment	Year 1	2	3	4	5
#1	10%	4%	12%	13%	6%
#2	6%	9%	6%	5%	9%
#3	17%	1%	11%	19%	2%

(r_{ik})

Annual Return

F = $10,000 available for investment

Let

x_i = amount of money to be allocated to investment #i

Assuming that past history is indicative of future performance, the expected annual return will be

$$E = \sum_{i=1}^{n} e_i x_i$$

where

$$e_i = \frac{1}{p}\sum_{k=1}^{p} r_{ik}$$

Expected Annual Returns

$$\begin{cases} e_1 = \frac{1}{5}(0.10 + 0.04 + 0.12 + 0.13 + 0.06) = 0.09 \\ e_2 = \frac{1}{5}(0.06 + 0.09 + 0.06 + 0.05 + 0.09) = 0.07 \\ e_3 = \frac{1}{5}(0.17 + 0.10 + 0.11 + 0.19 + 0.02) = 0.10 \end{cases}$$

The expected total annual return from the investments will be

$$E = 0.09\,x_1 + 0.07\,x_2 + 0.10\,x_3$$

$$E = 0.09\,x_1 + 0.07\,x_2 + 0.10\,x_3$$

If we wish to maximize the expected return, then we would invest the total available funds in investment #3, which has the highest expected return.
Looking at the past history, however, we see a greater variability in the return provided by investment #3:

#1	10%	4%	12%	13%	6%
#2	6%	9%	6%	5%	9%
#3	17%	1%	11%	19%	2%

The Variance of the total annual return, based upon past performance, is

$$V = \frac{1}{p}\sum_{k=1}^{p}\left[\sum_{i=1}^{n} r_{ik} x_i - E\right]^2$$

total return in year k if you had invested x

expected return per year

Substitute $E = \sum_{i=1}^{n} e_i x_i$ into $V = \frac{1}{p}\sum_k \left(\sum_i r_{ik} x_i - E\right)^2$

to get

$$V = \frac{1}{p}\sum_k \left(\sum_i r_{ik} x_i - \sum_i e_i x_i\right)^2$$
$$= \frac{1}{p}\sum_k \left(\sum_i [r_{ik} - e_i]\, x_i\right)^2$$
$$= \frac{1}{p}\sum_k \sum_i \sum_j (r_{ik} - e_i)(r_{jk} - e_j)\, x_i x_j$$

$$V = \sum_i \sum_j \sigma_{ij}^2 x_i x_j \quad \text{where} \quad \sigma_{ij}^2 = \frac{1}{p}\sum_k (r_{ik} - e_i)(r_{jk} - e_j)$$
$$= \frac{1}{p}\sum_k r_{ik} r_{jk} - \frac{1}{p^2}\left(\sum_k r_{ik}\right)\left(\sum_k r_{jk}\right)$$

Thus, the variance of the total return is the quadratic function $x^T C x$

where C is the covariance matrix with entries σ_{ij}^2

In our example, the covariance matrix is

$$C = \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} \times 10^{-4}$$

If we invest all $10,000 in #3 in order to maximize the expected return, i.e., $x = (0,0,10^4)$ the variance of the annual return will be

$$x^T C x = 55.2 \times 10^{-4} \times 10^4 \times 10^4 = 55.2 \times 10^4$$

i.e., the standard deviation will be $743, or 74% of the expected return ($1000)! ... a "risky" investment.

Suppose that we are "risk-averse" and are satisfied with an 8% annual rate of return, but wish to minimize the variance of the total return.

Then we must solve:

Quadratic Programming Problem

$$\text{Minimize } x^T \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} x$$

subject to

$$x_1 + x_2 + x_3 \le 10000$$

$$0.09x_1 + 0.07x_2 + 0.10\, x_3 \ge 800$$

$$x_j \ge 0,\ j=1,2,3$$

Optimal solution: $x = (5000,5000,0)$

i.e., invest half of the total in each of investments #1 & #2, and nothing in #3 which yields the greatest expected return!

The expected return will be 8% ($800), with a variance of 0.009×10^6, i.e., a standard deviation of $94.87.