

Annual Return

F = \$10,000 available for investment

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## Expected Annual Returns

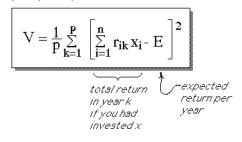
ĺ	$e_1 = \frac{1}{5}(0.10 + 0.04 + 0.12 + 0.13 + 0.06) = 0.09$
ł	$e_2 = \frac{1}{5} (0.06 + 0.09 + 0.06 + 0.05 + 0.09) = 0.07$
l	$e_3 = \frac{1}{5}(0.17 + 0.10 + 0.11 + 0.19 + 0.02) = 0.10$

The expected total annual return from the investments will be

$$E = 0.09 x_1 + 0.07 x_2 + 0.10 x_3$$

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The Variance of the total annual return, based upon past performance, is



A fixed sum of money F is to be allocated among n investments, each of which has a known history of returns during the previous p periods

- rik = return per dollar invested in investment #i during period k, i=1,...n; k=1,...p
- $\mathbf{X}_i$  = amount of money to be allocated to investment #i

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Let

X<sub>i</sub> = amount of money to be allocated to investment #i

Assuming that past history is indicative of future performance, the expected annual return will be

where

$$e_i = \frac{1}{p} \sum_{k=1}^p r_{ik}$$

 $\mathbf{E} = \sum_{i=1}^{n} \mathbf{e}_i \mathbf{x}_i$ 

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 $E=0.09\;x_1+0.07\;x_2+0.10\;x_3$ 

If we wish to maximize the expected return, then we would invest the total available funds in investment #3, which has the highest expected return.

Looking at the past history, however, we see a greater variability in the return provided by investment #3:

<b>#1</b> <b>#2</b>	10%	4% 9%	12%	13% 5%	6% 9%
#3	17%	1%	11%	19%	2%

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## $$\begin{split} \text{Substitute} \ \boxed{E = \sum_{i=1}^{n} e_i \, x_i} \quad \text{into} \quad \boxed{V = \frac{1}{p} \sum_k \left( \sum_i \, r_{ik} \, x_i - E \right)^2} \\ \text{to get } V = \frac{1}{p} \sum_k \left( \sum_i \, r_{ik} \, x_i \, -\sum_i \, e_i \, x_i \right)^2 \\ &= \frac{1}{p} \sum_k \left( \sum_i \, \left[ r_{ik} - e_i \right] \, x_i \right)^2 \\ &= \frac{1}{p} \sum_k \sum_i \sum_j \, (r_{ik} - e_i) \, (r_{jk} - e_j) \, x_i x_j \\ V = \sum_i \sum_j \sigma_{ij}^2 \, x_i x_j \quad \text{where} \quad \sigma_{ij}^2 = \frac{1}{p} \sum_k \, (r_{ik} - e_i) \, (r_{jk} - e_j) \\ &= \frac{1}{p} \sum_k r_{ik} r_{jk} - \frac{1}{p^2} \left( \sum_k \, r_{ik} \right) (\sum_k \, r_{jk}) \\ \end{split}$$

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Thus, the variance of the total return is the quadratic function  $\mathbf{x}^\mathsf{T} \subset \mathbf{x}$ 

where C is the covariance matrix with entries  $\sigma^2_{ii}$ 

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Suppose that we are "risk-averse" and are satisfied with an 8% annual rate of return, but wish to minimize the variance of the total return.

Then we must solve:

Quadratic Programming Problem  $\begin{array}{c|ccccc} \text{Minimize } x^{\mathsf{T}} & \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} x \\ \text{subject to} & \\ x_1 + x_2 + x_3 \leq 10000 \\ 0.09x_1 + 0.07x_2 + 0.10 & x_3 \geq 800 \\ x_j \geq 0, \ j=1,2,3 \end{array}$ 

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In our example, the covariance matrix is

$$C = \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} \times 10^{-4}$$

If we invest all \$10,000 in #3 in order to maximize the expected return, i.e.,  $x = (0,0,10^4)$ the variance of the annual return will be  $x^{T}C x = 55.2 \times 10^{-4} \times 10^4 \times 10^4 = 55.2 \times 10^4$ 

i.e., the standard deviation will be \$743, or 74% of the expected return (\$1000)! ... a "risky" investment.

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Optimal solution: x = (5000, 5000, 0)

i.e., invest half of the total in each of investments #1 & #2, and nothing in #3 which yields the greatest expected return!

The expected return will be 8% (\$800), with a variance of  $0.009 \times 10^6$ , i.e., a standard deviation of \$94.87.

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