the Poisson process

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The Poisson Process as a limiting case of the Bernoulli Process

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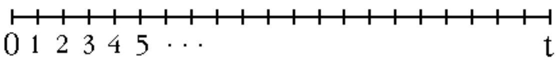
Consider the following situation:



A time interval of length t seconds is divided into one-second intervals, with the probability of a vehicle arriving at an intersection during a one-second interval being a small number p. (Assume that the probability that more than one vehicle arrives is negligible.)

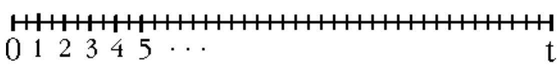
Consider the Bernoulli process $\{X_k ; k=1,2,...\}$ where $X_k = 1$ if a vehicle arrives during the k^{th} second, and the associated counting process $\{N_t\}$ which counts the number of arrivals during the interval $[0,t]$.

Then N_t has the binomial distribution:

$$P\{N_t = x\} = \binom{t}{x} p^x (1-p)^{n-x}$$

with expected value $\nu = tp$.

Consider what happens as we divide $[0,t]$ into n smaller time intervals, but in such as way that the expected number of arrivals in $[0,t]$ remains constant, ν.



That is, the probability of an arrival in each of these small intervals must be ν/n, and

$$P\{N_t = x\} = \binom{n}{x}\left(\frac{\nu}{n}\right)^x\left(1-\frac{\nu}{n}\right)^{n-x}$$

$$P\{N_t = x\} = \binom{n}{x}\left(\frac{\nu}{n}\right)^x\left(1-\frac{\nu}{n}\right)^{n-x}$$

$$= \left(\frac{n!}{x!(n-x)!}\right)\left(\frac{\nu^x}{n^x}\right)\left(1-\frac{\nu}{n}\right)^n\left(1-\frac{\nu}{n}\right)^{-x}$$

$$= \frac{\nu^x}{x!}\left(1-\frac{\nu}{n}\right)^n\left(\frac{n!}{(n-x)!}\right)\frac{1}{n^x\left(1-\nu/n\right)^x}$$

Consider the limit of this distribution as $n \longrightarrow +\infty$

$$P\{N_t = x\} = \frac{\nu^x}{x!}\underbrace{\left(1-\frac{\nu}{n}\right)^n}_{\downarrow\; e^{-\nu}}\underbrace{\left(\frac{n!}{(n-x)!}\right)\frac{1}{n^x\left(1-\nu/n\right)^x}}$$

$$\left(\frac{n!}{(n-x)!}\right)\frac{1}{n^x\left(1-\nu/n\right)^x} = \underbrace{\frac{n(n-1)(n-2)\ \ldots\ (n-x+1)}{\left[n\left(1-\nu/n\right)\right]^x}}_{\longrightarrow \frac{n^x}{n^x} = 1}$$

$$P\{N_t = x\} = \frac{\nu^x}{x!} e^{-\nu}$$

If the arrival rate is λ/second, then $\nu = \lambda t$ and

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

for x=0, 1, 2, 3, ...

Poisson Distribution

Poisson Distribution

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

for x=0, 1, 2, 3, ...

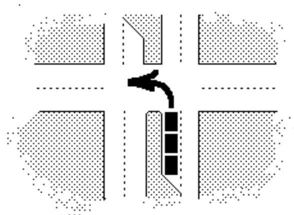
Mean Value $E(N_t) = \lambda t$

Variance $Var(N_t) = \lambda t$

mean and variance are equal!

Example A left-turn lane at an intersection has a capacity of 3 autos. 30% of autos arriving at the intersection wish to turn left.
The ***expected number*** of autos arriving during a red signal is 6.

What is the probability that the capacity of the left-turn lane is exceeded during a red signal?



Given that **N** autos arrive, the number **X** of left-turning autos has the ***binomial*** distribution.

The number **N** of autos arriving during the red signal has the ***Poisson*** distribution.

$$P\{X>3\} = \sum_{N=4}^{\infty} \underbrace{P\{X>3 \mid N \text{ arrivals}\}}_{\text{computed using binomial distn.}} \; \underbrace{P\{N \text{ arrivals}\}}_{\text{computed using Poisson distn.}}$$

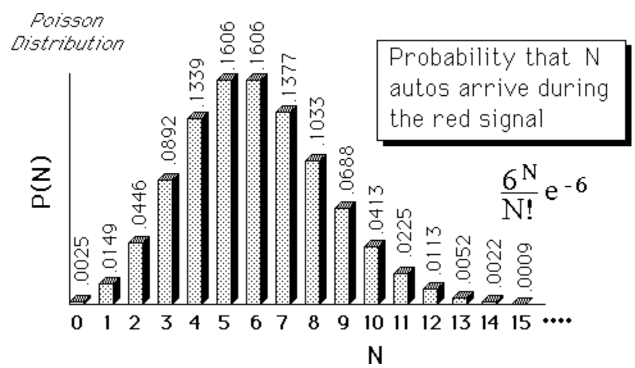
$$P\{X>3 \mid N \text{ arrivals}\} = \sum_{x=4}^{N} \binom{N}{x} (0.3)^x (0.7)^{N-x}$$

binomial distn.

$$= 1 - \sum_{x=0}^{3} \binom{N}{x} (0.3)^x (0.7)^{N-x}$$

$$P\{N \text{ arrivals}\} = \frac{6^N}{N!} e^{-6}$$

Poisson distn.



N	P{N}	P{X|N}	P{X|N}P{N}
0	0.00247875	0.00000000	0.00000000
1	0.01487251	0.00000000	0.00000000
2	0.04461754	0.00000000	0.00000000
3	0.08923508	0.00000000	0.00000000
4	0.13385262	0.00810000	0.00108421
5	0.16062314	0.03078000	0.00494398
6	0.16062314	0.07047000	0.01131911
7	0.13767698	0.12603600	0.01735226
8	0.10325773	0.19410435	0.02004278
9	0.06883849	0.27034090	0.01860986
10	0.04130309	0.35038928	0.01447216
11	0.02252896	0.43043766	0.00969731
12	0.01126448	0.50748423	0.00571655
13	0.00519899	0.57939435	0.00301227
14	0.00222814	0.64483257	0.00143678
15	0.00089126	0.70313207	0.00062667
⋮	⋮	⋮	⋮
			0.1083

The probability that the capacity of the left-turn lane is exceeded during each red signal is about 11%

Time between arrivals

Suppose that the number of arrivals in an interval has the Poisson distribution with arrival rate λ/second.

Let T_1 = time of the first arrival.
What is the distribution of T_1 ?

$P\{T_1 > t\} = P\{$ NO arrivals occur in interval $[0,t]\}$

$$= \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$$

← Poisson distribution $P\{N_t = 0\}$

CDF: $P\{T_1 \le t\} = F(t) = 1 - e^{-\lambda t}$

Density function: $f(t) = \frac{d}{dt} F(t) = \lambda e^{-\lambda t}$

Exponential Distribution

Exponential Distribution

$F(t) = 1 - e^{-\lambda t}$

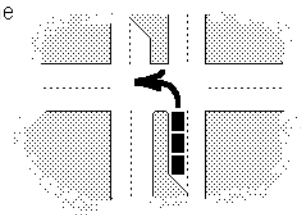
Mean Value: $E(T_1) = \frac{1}{\lambda}$

Variance: $Var(T_1) = \frac{1}{\lambda^2}$

Example

Suppose that the arrival rate for northbound autos is 6 per 30 second red signal, i.e., 0.2/second

What is the distribution of the arrival time of the first auto?
(This will also be the distribution of the time between arrivals!)



Exponential Distribution

$f(t) = \lambda e^{-\lambda t}$

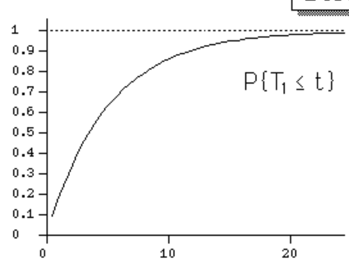
$\lambda = 0.2$/sec.

Exponential Distribution

$\lambda = 0.2$/sec.

t	F(t)
1	0.18127
2	0.32968
3	0.45119
4	0.55067
5	0.63212
6	0.69881
7	0.75340
8	0.79810
9	0.83470
10	0.86466
11	0.88920
12	0.90928
13	0.92573
14	0.93919
15	0.95021
16	0.95924
17	0.96663
18	0.97268
19	0.97763
20	0.98168

$F(t) = 1 - e^{-\lambda t}$



Memoryless Property — **Exponential Distribution**

Suppose that it is known that, at time t_0, the first arrival has not yet occurred, i.e., $T_1 > t_0$.

What is the conditional distribution of T_1 ?

That is, what is $P\{T_1 \le t \mid T_1 > t_0\}$ for $t > t_0$?

Memoryless Property — **Exponential Distribution**

$$P\{T_1 \le t \mid T_1 > t_0\} = \frac{P\{T_1 \le t \cap T_1 > t_0\}}{P\{T_1 > t_0\}} = \frac{P\{t_0 \le T_1 \le t\}}{P\{T_1 > t_0\}}$$

$$= \frac{F(t) - F(t_0)}{1 - F(t_0)} = \frac{(1 - e^{-\lambda t}) - (1 - e^{-\lambda t_0})}{e^{-\lambda t_0}}$$

$$= \frac{e^{-\lambda t_0} - e^{-\lambda t}}{e^{-\lambda t_0}} = 1 - e^{-\lambda (t - t_0)}$$

Memoryless Property — **Exponential Distribution**

$P\{T_1 \le t \mid T_1 > t_0\} = 1 - e^{-\lambda (t - t_0)} = P\{T_1 \le t - t_0\}$

If the time τ is reckoned from time t_0, i.e., $\tau = t - t_0$, then

$P\{T_1 \le t \mid T_1 > t_0\} = P\{T_1 \le t - t_0\} = P\{T_1 \le \tau\}$

In other words, the failure of an arrival to occur before time t_0 does not alter one's prediction of the length of time (from t_0) before the next arrival.

Time of k^{th} Arrival

Let T_k = time of k^{th} arrival,
$\tau_k = T_k - T_{k-1}$ = time between arrivals k-1 and k.

Suppose that τ_k (k=1,2,3,....) have identical and independent exponential distributions with rate .
Then T_k is the *sum* of k random variables with exponential distributions.
It is said to have a *k-Erlang* distribution.

Erlang Distribution

time of k^{th} arrival in a Poisson process

Density function

(k>0, λ>0, t≥0)

$$f(t) = \frac{\lambda (\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} = \frac{\lambda (\lambda t)^{k-1} e^{-\lambda t}}{\Gamma(k)}$$

where the *Gamma* function is defined by
(for k>0, not necessarily integer!)

$$\Gamma(k) = \int_0^{\infty} e^{-u} u^{k-1}\, du$$

$= (k-1)!$ if k integer

Erlang Distribution

CDF

$$F(t) = \frac{\Gamma(k, \lambda t)}{\Gamma(k)}$$

where $\Gamma(k, x)$ is the *"incomplete Gamma function"* defined by

$$\Gamma(k, x) = \int_0^{x} e^{-u} u^{k-1}\, du$$

tabulated

Alternate computation, when k is integer

CDF

$$F(t) = P\{T_k \le t\} = P\{N_t \ge k\}$$
$$= 1 - P\{N_t < k\}$$
$$= 1 - P\{N_t \le k-1\}$$

where N_t = # arrivals at time t has the *Poisson* distribution:

$$F(t) = 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

Erlang Distribution

Mean Value $\mu = \dfrac{k}{\lambda}$

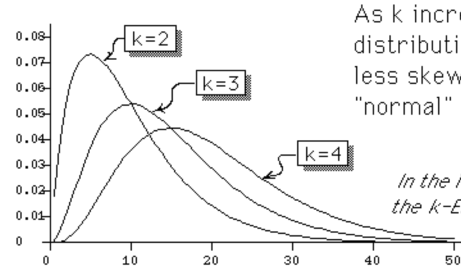
Variance $\sigma^2 = \dfrac{k}{\lambda^2}$

(These expressions result from the fact that the random variable is the sum of k i.i.d. random variables.)

More generally, when k is not an integer, the probability distribution is called the *Gamma* distribution.

Erlang Distribution

example: $\lambda = 0.2$



As k increases, the distribution becomes less skewed, and more "normal"

In the limit, as k → ∞, the k-Erlang distribution converges to the Normal distribution!

Bernouilli process		Poisson process
Binomial distn.	# of events	Poisson distn.
Geometric distn.	time until 1st event	Exponential distn.
Pascal distn.	time until k^{th} event	Erlang distn.