Markov Chain Examples

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dennis-bricker@uiowa.edu

author

- ☞ Allcity Auto Insurance
- ☞ Airline credit cardholders
- ☞ Airline departures
- ☞ Airline reservation system
- ☞ Refrigerator warranty
- ☞ Used textbook sales
- ☞ "Passing the buck"

Example

Allcity Insurance Co. sets auto insurance premiums based upon a customer's accident history.

Premium	Accident History
$100	no accident during past 2 years
$400	accident during each of last 2 yrs.
$300	accident during only 1 of last 2 yrs.

A customer who has had an accident during the last year has a 10% chance of having an accident during the current year.

If he/she has not had an accident during the last year, there is only a 3% chance of having an accident during the current year.

During a typical year, what is the average premium paid by an Allcity customer?

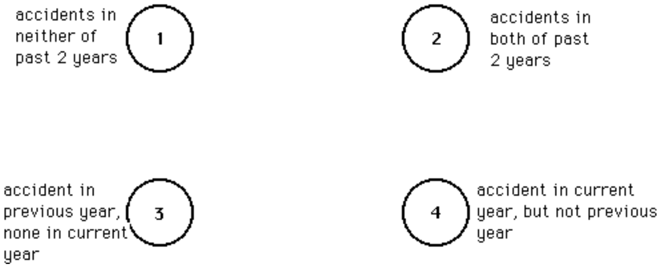
Model

States of System

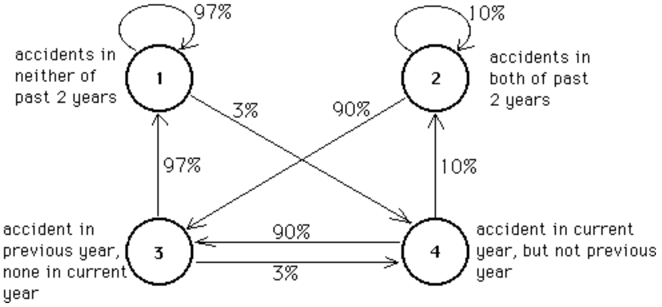
At the end of each year, the policyholders will be determined to be in one of the 4 states:

1. No accidents during current & previous years
2. Accidents during both current & previous years
3. No accident in current year, but accident in the previous year
4. Accident in current year, but no accident in the previous year

Transitions between states



Transitions between states



```
    Auto Insurance Policy
    ---------------------

Transition Probability Matrix
-----------------------------
    f|
    r|
    o|   1    2   3    4
    m| ---- --- --- ----
    1| 0.97 0   0   0.03
    2| 0    0.1 0.9 0
    3| 0.97 0   0   0.03
    4| 0    0.1 0.9 0
```

```
Steady State Distribution
-------------------------

   i P{i}
   1 0.93870968
   2 0.0032258065
   3 0.029032258
   4 0.029032258
```

```
           Auto Insurance Policy
           ---------------------

           Mean First Passage Times
           ------------------------
f|                 to
r|                 --
o|    1         2          3          4
m|--------- --------- ---------- ---------
1|1.0652921 343.33333 34.444444  33.333333
2|2.1764032 310        1.1111111 34.444444
3|1.0652921 343.33333 34.444444  33.333333
4|2.1764032 310        1.1111111 34.444444
```

```
              Auto Insurance Policy
              ---------------------

          i      Pi        C     Pi×C
          ------------    ---  ----------
          1 0.93870968    100 93.870968
          2 0.0032258065  400  1.2903226
          3 0.029032258   300  8.7096774
          4 0.029032258   300  8.7096774

 The average cost/period in steady state is 112.58065
```

```
                 Auto Insurance Policy
                 ---------------------

Present value of all future costs, for each initial state:
(using interest rate 10%, i.e. discount factor 0.90909091)

                   i         PV
                   -     ---------
                   1     1219.4175
                   2     1717.4757
                   3     1419.4175
                   4     1617.4757
```

Example

A recently-completed survey of subscribers to a travel magazine shows that 65% of them have at least one airline credit card. When compared to a similar survey taken 5 years ago, the data indicate that 40% of those individuals who did not have an airline credit card subsequently have obtained one, while 10% of those who carried such cards 5 years ago no longer do so.

Assuming these trends continue, what fraction of subscribers will own airline credit cards
(i) 10 years from now?
(ii) over the long run? ⏎

Example

An airline with a 7:15 am commuter flight between NY and DC does not want the flight to depart late 2 days in a row. If the flight leaves late one day, the airline makes a special effort to have the flight leave on time, and succeeds 90% of the time.
If the flight was not late in leaving the previous day, the airline makes no special effort, and the flight departs as scheduled 60% of the time.

What percentage of the time is the flight late in departing? How frequently do they depart late 2 days in a row? ⏎

Example

An airline reservation system has 2 computers.

- only one is in operation at any given time, while the other is used as a back-up.
- a computer may break down on any given day with probability p= 5%
- there is a single repair facility, which takes 2 days to restore a computer to normal. The repair facility can deal with only one computer at a time.

What is the average time between system failures, i.e., days in which neither computer is functioning? ⏎

Model

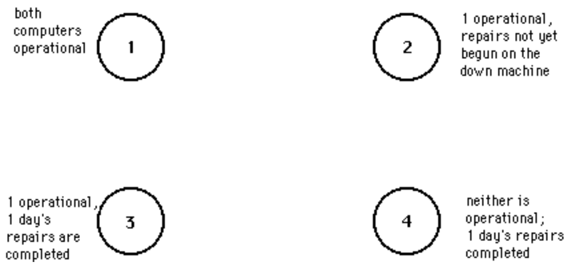
Assume that the system is observed at the end of each day, and that, to simplify the model,

- any computer failures occur at the end of the day
- both repair of the failed machine and the beginning of use of the back-up machine begins the next morning

States of System

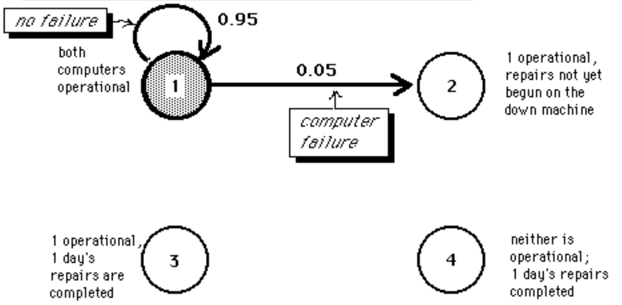
1. Both computers are operational
2. Only one computer is operational, with the first machine having failed during the current day (so that repair has not yet begun)
3. Only one computer is operational, and the first day of repairs has been completed on the failed machine
4. Neither computer is operational: the first day of repairs has been completed on the first machine to go down.

Transitions between states



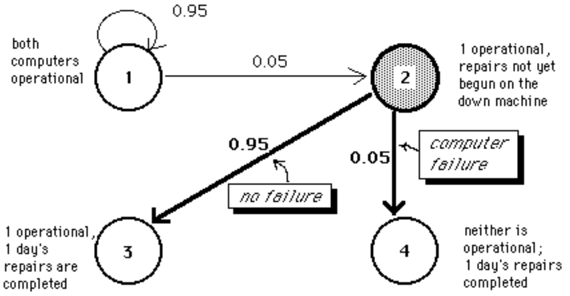
Transitions between states

FROM STATE 1



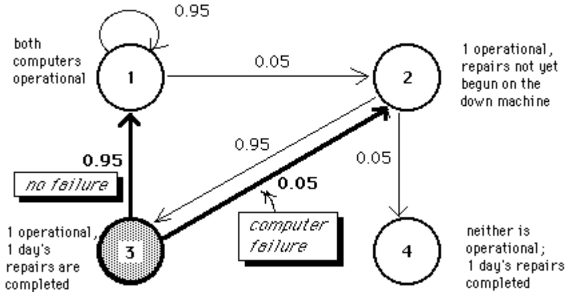
Transitions between states

FROM STATE 2



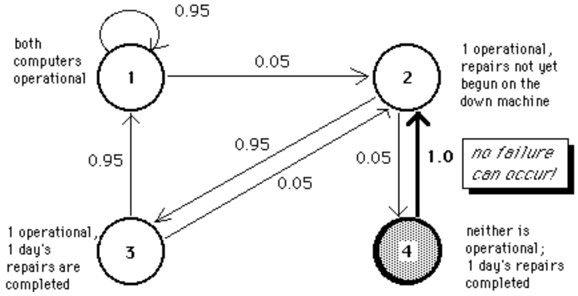
Transitions between states

FROM STATE 3

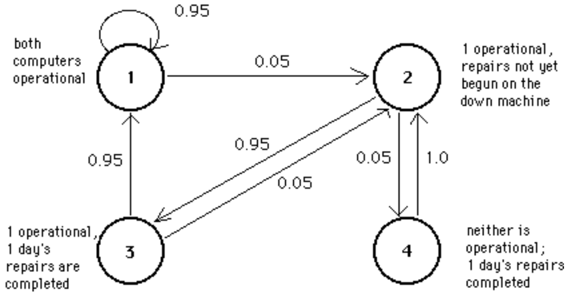


Transitions between states

FROM STATE 4



Transitions between states



Airline Reservation System

Transition Probability Matrix

to	1	2	3	4
f 1	0.95	0.05	0	0
r 2	0	0	0.95	0.05
o 3	0.95	0.05	0	0
m 4	0	1	0	0

Steady State Distribution

i	P{i}
1	0.90024938
2	0.049875312
3	0.047381546
4	0.0024937656

Airline Reservation System

Mean First Passage Times

to	1	2	3	4
f 1	1.1108033	20	21.105263	420
r 2	2.2160665	20.05	1.1052632	400
o 3	1.1108033	20	21.105263	420
m 4	3.2160665	1	2.1052632	401

First-Passage Probabilities to State 4 from State 1

n	$f_{14}^{(n)}$
1	0
2	0.0025
3	0.002375
4	0.002375
5	0.0023690625
6	0.0023634219
7	0.0023577813
8	0.0023521547
9	0.0023465416
10	0.0023409419
11	0.0023353555
12	0.0023297825
13	0.0023242227
14	0.0023186763
15	0.002313143
sum=	0.033001084

Example

Coldspot manufactures refrigerators. The company has issued a warranty on all refrigerators that requires free replacement of any refrigerator that fails before it is 3 years old.

- 3% of all new refrigerators fail during their first year of operation
- 5% of all 1-year-old refrigerators fail during their second year of operation
- 7% of all 2-year-old refrigerators fail during their third year of operation

Replacement refrigerators are not covered by the warranty.

What fraction of the refrigerators will Coldspot have to replace?

Suppose that it costs Coldspot $500 to replace a refrigerator, and that the company sells 10,000 units per year. How much would be saved in replacement costs by reducing the warranty period to 2 years?

Example

The best-selling textbook ,*The Joy of O.R.*, sells 100,000 copies every fall. Some users of the book will keep it, and some sell the book back to the bookstore.

- 90% of all students who buy a new book sell it back, 80% of all who buy a once-used book sell it back, and 60% of all who buy a twice-used book sell it back.
- If a book has been used 4 or more times, the cover falls off, and it cannot be sold back.

The bookstore's profit on each type of book is

New:	$6.00
Once-used:	$3.00
Twice-used:	$2.00
Thrice-used:	$1.00

- ***In the steady state, how many new copies of the textbook will the publisher be able to sell each year?***
- ***What will be the bookstore's average profit per book?***
- ***What is the average number of years that a textbook circulates?***
- ***What is the average age of a circulating textbook?***

Model

If we define the "system" to be a single book, with an absorbing state representing the book being out of circulation, then we cannot easily answer the question concerning the average profit per book sold by the bookstore.

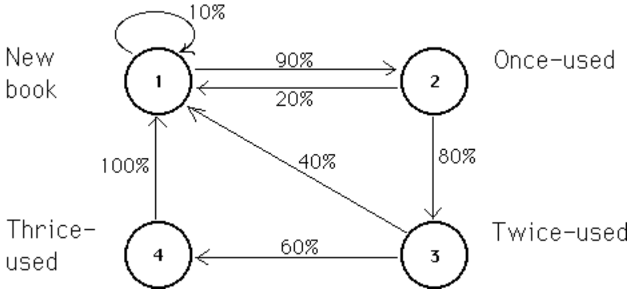
Instead, we define the "system" to be a unit of inventory of the text on the bookstore shelf, with books taken out of circulation being replaced by new books from the publisher.

States of System

(observed at the beginning of each semester in which the book is used.)

1. New book
2. Once-used book
3. Twice-used book
4. Thrice-used book

Transitions between states



```
Textbook Sales


Transition Probability Matrix

f|
r|
o|  1   2   3   4
m| --- --- --- ---
1| 0.1 0.9 0   0
2| 0.2 0   0.8 0
3| 0.4 0   0   0.6
4| 1   0   0   0
```

```
Textbook Sales


System costs/revenues by state


|i| name       | --COST--
|1| New book   | 6
|2| Once-used  | 3
|3| Twice-used | 2
|4| Thrice-used| 1
```

```
Textbook Sales


 i State               Pi       C    Pi×C

|1| New book    | 0.327654   6 1.965924
|2| Once-used   | 0.2948886  3 0.88466579
|3| Twice-used  | 0.23591088 2 0.47182176
|4| Thrice-used| 0.14154653 1 0.14154653

The average cost/period in steady state is 3.4639581
```

```
                 Textbook Sales
Initial state: 1, interest rate= 15%
              Present Values by Period
   n        Cn              PVCn           Vn

   0     6               6                 6
   1     3.3             2.8695652         8.8695652
   2     2.85            2.1550095        11.024575
   3     3.039           1.9981918        13.022766
   4     4.3593          2.4924439        15.51521
   5     3.22935         1.6055577        17.120768
   6     3.214041        1.3895186        18.510287
   7     3.4710135       1.3048825        19.815169
   8     3.7388991       1.2222528        21.037422
   9     3.3193954       0.94357933       21.981001
  10     3.393059        0.83871229       22.819714
```

Expected profit in yr. n

Present value

Cumulative present value

Textbook Sales

Mean First Passage Times

```
f|                to
r|                --
o|  1        2         3         4
m|----- --------- --------- ---------
1|3.052 1.1111111 2.6388889 6.0648148
2|2.28  3.3911111 1.5277778 4.9537037
3|1.6   2.7111111 4.2388889 3.4259259
4|1     2.1111111 3.6388889 7.0648148
```

i	name
1	New book
2	Once-used
3	Twice-used
4	Thrice-used

The *Herald Tribune* has obtained the following information about its subscribers:

- during the 1st year as subscribers, 20% will cancel their subscriptions
- of those who have subscribed for 1 year, 10% will cancel their subscriptions in the 2nd year
- of those who have been subscribing more than 2 years, 4% will cancel during any given year.

On the average, how long does a subscriber subscribe to this newspaper?

General Motors has 3 auto divisions (A,B,&C), served by an accounting & management consulting departments.

They desire to know what % of cost of the support departments should be allocated to each of the auto divisions.

The following table gives the fraction of the support given to each of the divisions

	Acctg.	Mgmt consult.	Auto A	Auto B	Auto C
Acctg.	10%	30%	20%	20%	20%
Mgmt consult.	30%	20%	30%	0	20%

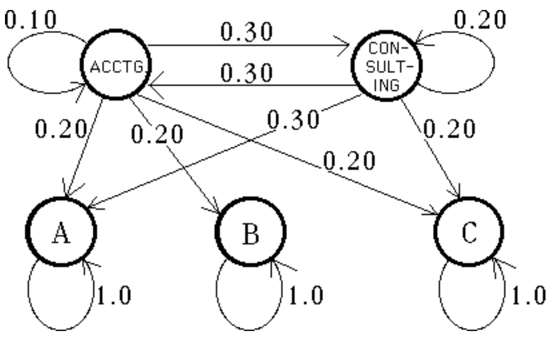
Each year, it costs $63 million to run the Accounting Dept., and $210 million to run the Mgmt. Consulting Dept. These costs are to be allocated entirely to the auto divisions. How should this be done?

Model

We will model a dollar of cost as a Markov chain, with 5 possible states:

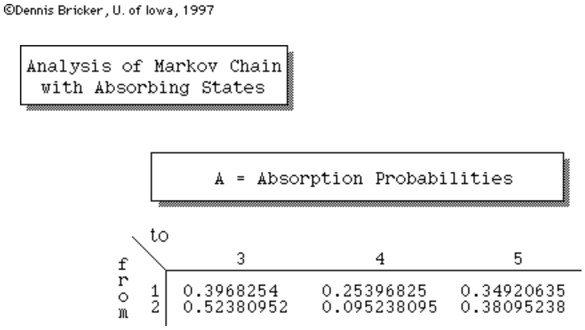
1) Accounting Dept.
2) Management Consulting Dept.
3) Auto Division A
4) Auto Division B
5) Auto Division C

This dollar gets "passed" from one support department to another until eventually it gets "absorbed" by one of the 3 auto divisions!



Transition Probability Matrix

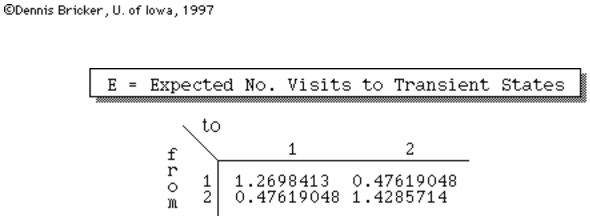
```
     to
       1   2   3   4   5
f  1 0.1 0.3 0.2 0.2 0.2
r  2 0.3 0.2 0.3 0   0.2
o  3 0   0   1   0   0
m  4 0   0   0   1   0
   5 0   0   0   0   1
```



Analysis of Markov Chain with Absorbing States

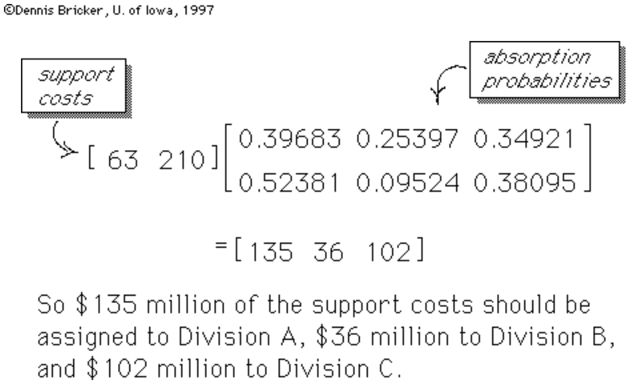
A = Absorption Probabilities

from \ to	3	4	5
1	0.3968254	0.25396825	0.34920635
2	0.52380952	0.095238095	0.38095238



E = Expected No. Visits to Transient States

from \ to	1	2
1	1.2698413	0.47619048
2	0.47619048	1.4285714



support costs

absorption probabilities

$$[\,63\quad 210\,]\begin{bmatrix}0.39683 & 0.25397 & 0.34921\\ 0.52381 & 0.09524 & 0.38095\end{bmatrix}$$

$$=[\,135\quad 36\quad 102\,]$$

So \$135 million of the support costs should be assigned to Division A, \$36 million to Division B, and \$102 million to Division C.