Facility Location Problem in a Network

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☞ Median Problem

minimizing the sum of weighted shortest path lengths

☞ Center Problem

minimizing the maximum of (possibly) weighted shortest path lengths

The p-Median Problem

Given a network with nodes j=1,2,...n
where w_j = "weight" of node j
(e.g., volume of shipments)
Let d(X,j) = distance from node j to the nearest point in the set X
Find $X = \{x_1, x_2, \dots x_p\}$ which

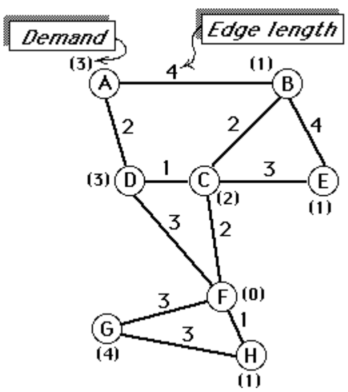
minimizes $\tau(X) = \sum_{j=1}^{n} w_j d(X,j)$

The points in X are called p-medians.

Hakimi's Theorem

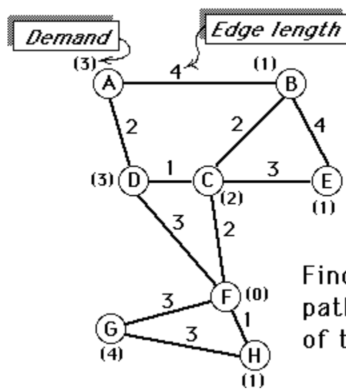
At least one set of p-medians exist solely on the nodes of the network.

That is, we need search only among the nodes for the p-medians!



Where should a single facility be located to serve the eight cities?

*Objective:
Minimize the sum of the distances to the cities weighted by their demands*



TO / FROM

	A	B	C	D	E	F	G	H
A	0	4	3	2	6	5	8	6
B	4	0	2	3	4	4	7	5
C	3	2	0	1	3	2	5	3
D	2	3	1	0	4	3	6	4
E	6	4	3	4	0	5	8	6
F	5	4	2	3	5	0	3	1
G	8	7	5	6	8	3	0	3
H	6	5	3	4	6	1	3	0

Find the matrix of shortest path lengths between nodes of the network (e.g., by Floyd's algorithm)

TO / FROM

	A	B	C	D	E	F	G	H
A	0	4	3	2	6	5	8	6
B	4	0	2	3	4	4	7	5
C	3	2	0	1	3	2	5	3
D	2	3	1	0	4	3	6	4
E	6	4	3	4	0	5	8	6
F	5	4	2	3	5	0	3	1
G	8	7	5	6	8	3	0	3
H	6	5	3	4	6	1	3	0
W_j	3	1	2	3	1	0	4	1

shortest paths

TO / FROM

	A	B	C	D	E	F	G	H	$\sum_j W_j d_{ij}$
A	0	4	6	6	6	0	32	6	60
B	12	0	4	9	4	0	28	5	62
C	9	2	0	3	3	0	20	3	40
D	6	3	2	0	4	0	24	4	43
E	18	4	6	12	0	0	32	6	78
F	15	4	4	9	5	0	12	1	50
G	24	7	10	18	8	0	0	3	70
H	18	5	6	12	6	0	12	0	59

weighted shortest paths

$\sum_j W_j d_{ij}$

TO / FROM

	A	B	C	D	E	F	G	H	$\sum_j W_j d_{ij}$
A	0	4	6	6	6	0	32	6	60
B	12	0	4	9	4	0	28	5	62
C	9	2	0	3	3	0	20	3	40
D	6	3	2	0	4	0	24	4	43
E	18	4	6	12	0	0	32	6	78
F	15	4	4	9	5	0	12	1	50
G	24	7	10	18	8	0	0	3	70
H	18	5	6	12	6	0	12	0	59

weighted shortest paths

$\sum_j W_j d_{ij}$

☜ Minimum

The optimal location for a single facility to serve the 8 cities is at city C

What if two facilities were to be used?

Consider all pairs of potential facility sites:

Examples: *select minimum shipping cost in each column*

A	0	4	6	6	6	0	32	6
B	12	0	4	9	4	0	28	5

$47 = \sum_j \underset{i=A,B}{\text{minimum}} \{W_j d_{ij}\}$

D	6	3	2	0	4	0	24	4
E	18	4	6	12	0	0	32	6

$39 = \sum_j \underset{i=D,E}{\text{minimum}} \{W_j d_{ij}\}$

A	0	4	6	6	6	0	32	6
G	24	7	10	18	8	0	0	3

$25 = \sum_j \underset{i=A,G}{\text{minimum}} \{W_j d_{ij}\}$

There are $\binom{8}{2} = 28$ such combinations to evaluate!

How might one find the 3-median set?

Requires considering $\binom{8}{3} = 56$ combinations!

A	0	4	6	6	6	0	32	6
B	12	0	4	9	4	0	28	5
C	9	2	0	3	3	0	20	3

$29 = \sum_j \underset{i=A,B,C}{\text{minimum}} \{W_j d_{ij}\}$

A	0	4	6	6	6	0	32	6
B	12	0	4	9	4	0	28	5
D	6	3	2	0	4	0	24	4

$34 = \sum_j \underset{i=A,B,D}{\text{minimum}} \{W_j d_{ij}\}$

etc.

APL evaluation of $\sum_j \underset{i \in S}{\text{minimum}} \{W_j d_{ij}\}$

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Math Programming Model of the p-Median Problem

Variables

X_{ij} = fraction of demand of customer j supplied by facility at location i

$$Y_i = \begin{cases} 1 \text{ if a facility is located at site i} \\ 0 \text{ otherwise} \end{cases}$$

Math Programming Model of the p-Median Problem

$$\text{Min} \sum_{i=1}^{m} \sum_{j=1}^{n} W_j D_{ij} X_{ij}$$

subject to

$$\sum_{i=1}^{m} X_{ij} = 1 \quad \forall j=1,\ldots n$$

$$X_{ij} \le Y_i \quad \forall\, i=1,\ldots m; j=1,\ldots n$$

$$\sum_{i=1}^{m} Y_i = p$$

$$X_{ij} \ge 0 \quad \forall\, i=1,\ldots m; j=1,\ldots n$$

$$Y_i \in \{0,1\} \quad \forall\, i=1,\ldots m$$

Heuristic Algorithm for the p-Median Problem

1. Initialization:
Let k=1. Find the 1-median (the set $S=X_1$)

2. Facility Addition:
Evaluate the (n−k) combinations of S with a node r not in S, i.e.,

$$\sum_j \underset{i \in S \cup \{r\}}{\text{minimum}} \{W_j d_{ij}\} \quad \forall\, r \notin S$$

Add to S the node yielding the lowest objective function and set k=k+1.

3. Facility Substitution:
Evaluate each of the kx(n−k) sets obtained by substituting a node not in S for a node in S, i.e.

$$\sum_j \underset{i \in S \cup \{r\} \setminus \{s\}}{\text{minimum}} \{W_j d_{ij}\} \quad \forall\, r \notin S \;\&\; s \in S$$

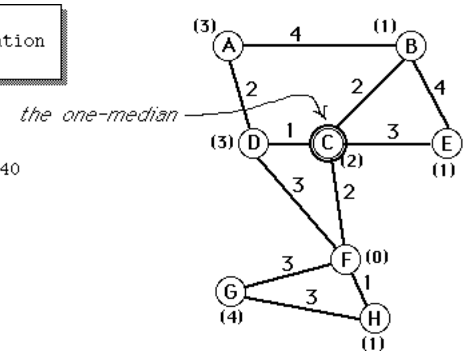
Replace S by the best set evaluated.

4. If S contains p nodes, i.e., k=p, STOP. Otherwise, return to step 2.

K-median Facility Location Problem

1-Median

Cost = 40



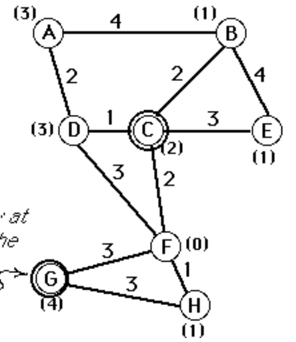
... beginning with 1-median set {C}

2-Median

Trial additions:

Add	1	2	4	5	6	7	8
cost	31	38	34	37	30	20	29

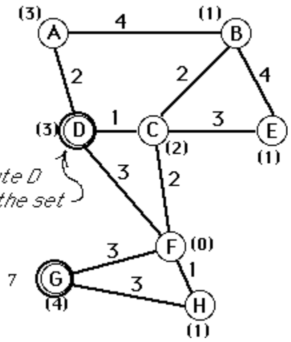
Addition result: Locations 3 7
Cost: 20



Substitution Step

Cost	Locations
25	1 7
31	3 1
32	2 7
38	3 2
18	4 7
34	3 4
43	5 7
37	3 5
38	6 7
30	3 6
47	8 7
29	3 8

Substitution result: Locations 4 7
Cost: 18



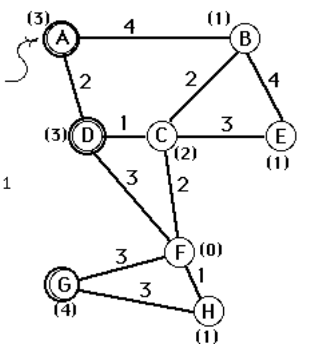
... begin with D & G in set

3-Median

Trial additions:

Add	1	2	3	5	6	8
cost	12	15	14	14	16	15

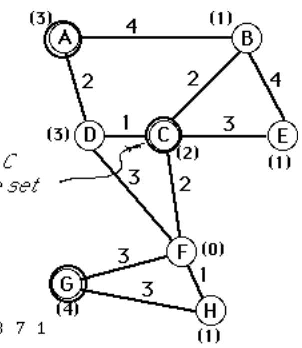
Addition result: Locations 4 7 1
Cost: 12



Substitution Step

Cost	Locations
17	2 7 1
34	4 2 1
15	4 7 2
11	3 7 1
28	4 3 1
14	4 7 3
19	5 7 1
33	4 5 1
14	4 7 5
20	6 7 1
22	4 6 1
16	4 7 6
22	8 7 1
21	4 8 1
15	4 7 8

Substitution result: Locations 3 7 1
Cost: 11



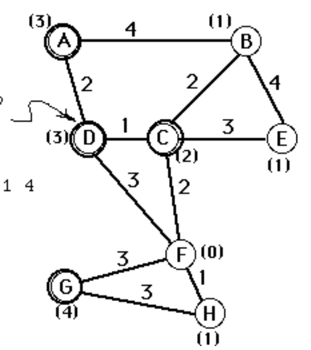
... begin with A, C, & G in set

4-Median

Trial additions:

Add	2	4	5	6	8
cost	9	8	8	9	8

Addition result: Locations 3 7 1 4
Cost: 8

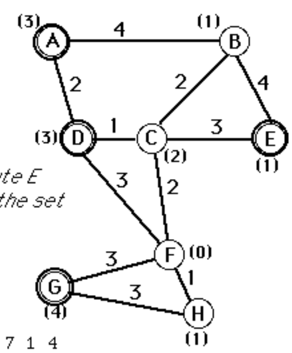


begin with A, C, D, & G (1,3,4,7)

Substitution Step

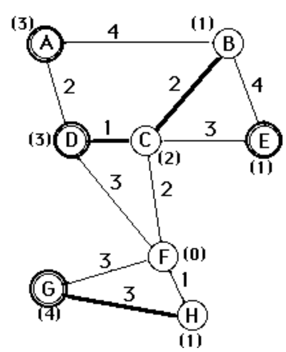
Cost	Locations
9	2 7 1 4
26	3 2 1 4
12	3 7 2 4
9	3 7 1 2
8	5 7 1 4
25	3 5 1 4
11	3 7 5 4
8	3 7 1 5
10	6 7 1 4
18	3 6 1 4
12	3 7 6 4
9	3 7 1 6
9	8 7 1 4
17	3 8 1 4
11	3 7 8 4
8	3 7 1 8

Substitution result: Locations 5 7 1 4
Cost: 8



Allocation of Customers to Warehouses

FROM	A	B	C	D	E	F	G	H
A	0	4	6	6	6	0	32	6
D	6	3	2	0	4	0	24	4
E	18	4	6	12	0	0	32	6
G	24	7	10	18	8	0	0	3



1-Median of a Tree

For any set C of vertices, define $W(C) = \sum_{i \in C} w_i$

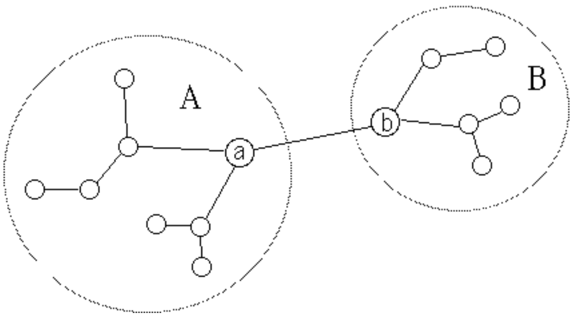
Theorem: Let [a,b] be any edge of a tree, and

let A = set of vertices reachable from a without passing through b

B = set of vertices reachable from b without passing through a.

Then $W(A) \geq W(B)$ implies $\tau(a) \leq \tau(b)$

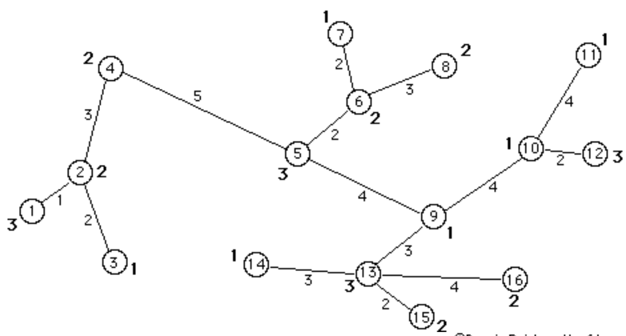
$W(A) \geq W(B)$ implies $\tau(a) \leq \tau(b)$



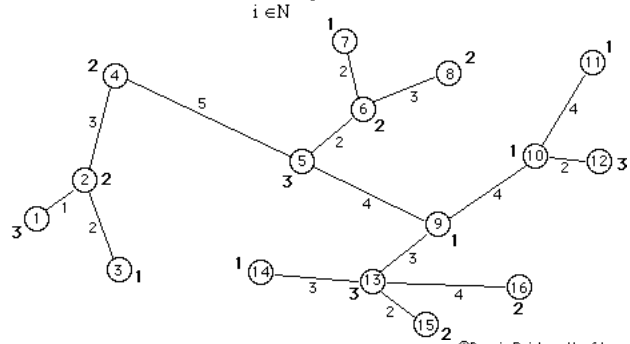
To find the 1-median of a tree:

0. Let $W = \sum_{i \in N} w_i$. Select any vertex j.
1. If $w_j \geq \frac{1}{2} W$, then stop; j is a 1-median.
2. If j has degree 1, let k be its neighbor, i.e., [k,j] will be an edge.
   Replace w_k with $w_k + w_j$, and delete vertex j from the tree.
   Else find an elementary chain from vertex j to a vertex k with degree 1 (preferably using previously unused edges.)
   Let j=k and return to step 1.

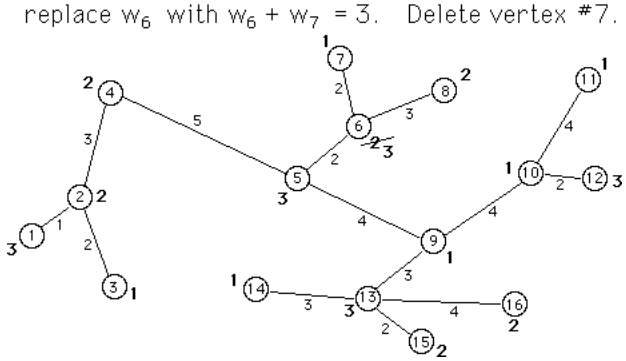
Example Find the 1-median of the tree:



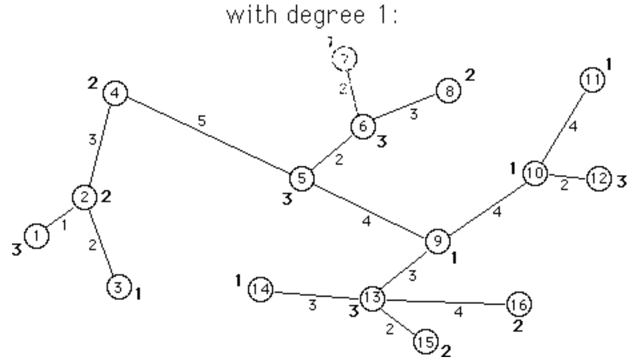
Let's choose to begin with vertex #7.
Total "demand" $W = \sum_{i \in N} w_i$ is 30.



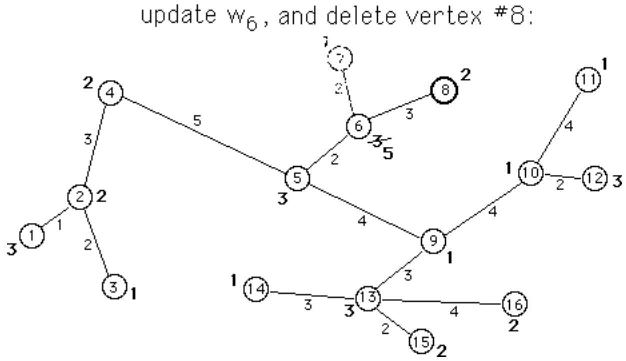
$w_7 < W/2 = 15$. Select neighbor (vertex #6), and replace w_6 with $w_6 + w_7 = 3$. Delete vertex #7.



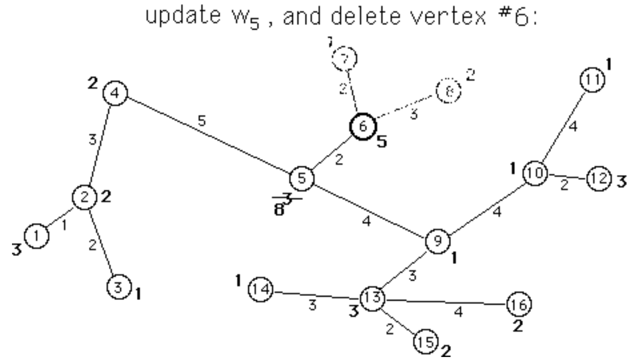
$w_6 < W/2 = 15$. Find a path 6->8 to a vertex (#8) with degree 1:



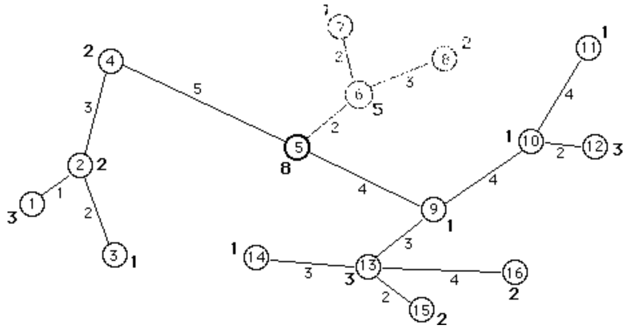
$w_8 < W/2 = 15$. Select neighbor (vertex #6), update w_6, and delete vertex #8:



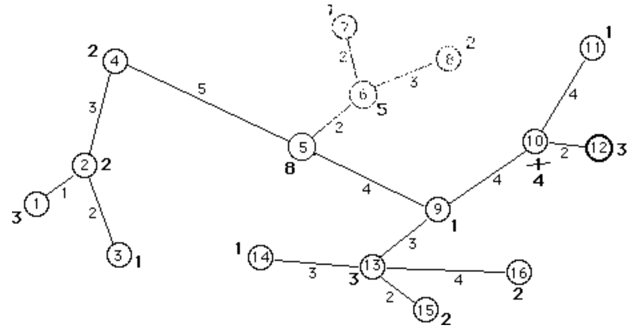
$w_6 < W/2 = 15$. Select neighbor (vertex #5), update w_5, and delete vertex #6:



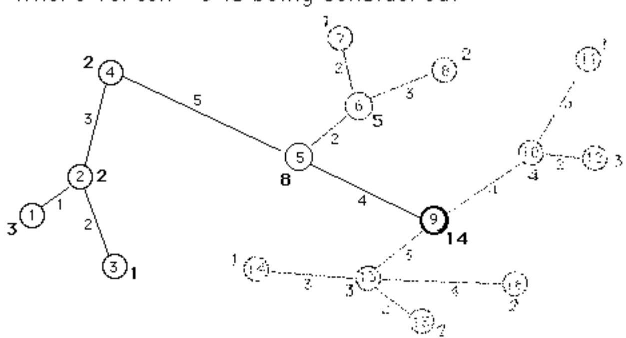
$w_5 < W/2 = 15$. Select chain 5→9→10→12 to vertex #12, which has degree 1.



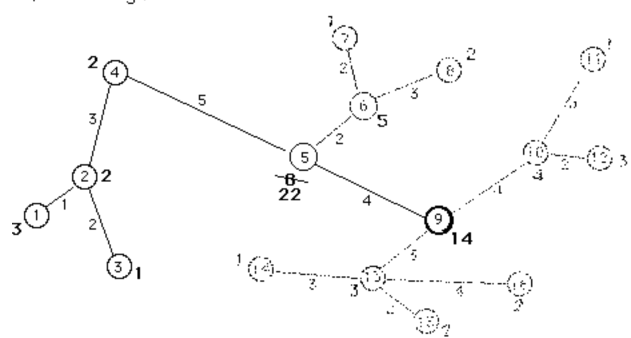
$w_{12} < W/2 = 15$. Select neighbor (vertex #10), update w_{10}, and delete vertex #12



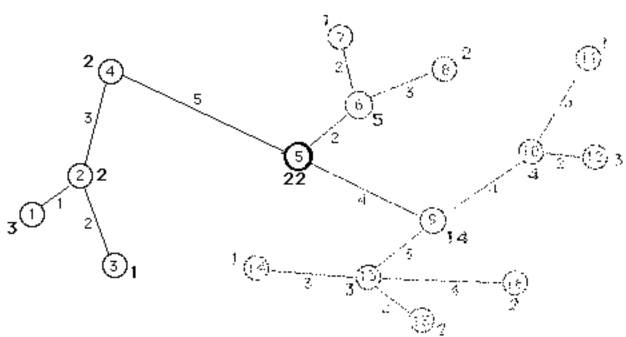
... after several more iterations, the tree is as shown, where vertex #9 is being considered.



$w_9 < W/2 = 15$, so we select its neighbor (vertex #5), update w_5, and delete vertex #9.

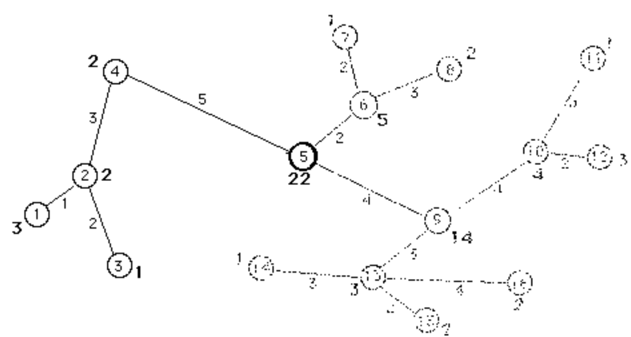


$w_5 > W/2 = 15$, so we stop; Vertex #5 is the 1-median.



$W(A) \geq W(B)$ implies $\tau(a) \leq \tau(b)$

Edge (5,9): $W(9) = 14 < 16 = W(5)$ implies $\tau(9) > \tau(5)$



$W(A) \geq W(B)$ implies $\tau(a) \leq \tau(b)$

Edge [5,4]: $W(5) = 22 > 8 = W(4)$ implies $\tau(4) > \tau(5)$