Machine Replacement Problem
Markov Decision Model

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Machine Replacement Problem

At the beginning of each month, a machine is inspected and classified as:

1) Good as new
2) Operable, with minor deterioration
3) Operable, with major deterioration
4) Inoperable

Machine Replacement Problem

After determining the state of the machine, a decision must be made:

1) Keep the machine another month
2) Replace the machine with a new machine

Machine Replacement Problem

A replacement machine costs $3000, minus trade-in value:

$1000 if in state 2
500 if in state 3
0 if in state 4

Monthly operating costs are $100, $200, and $500 for a machine in states 1, 2, & 3, respectively.

Machine Replacement Problem

Survival probabilities

from: State \ to: State	1	2	3	4
1	0.75	0.1875	0.0625	0
2	—	0.75	0.1875	0.0625
3	—	—	0.75	0.25

What is the optimal replacement policy?

States

i	name
1	Good as new
2	Minor deterioration
3	Major deterioration
4	Failed

Actions

k	name
1	Keep
2	Replace

Cost Matrix

k	name	1	2	3	4
1	Keep	100	200	500	9999
2	Replace	9999	2100	2600	3100

(Rows ~ actions, Columns ~ states)

A value of 9999 above signals
an infeasible action in a state.

(includes cost of operating the new machine, if decision is to replace)

Transition Probabilities

Action: Keep

from \ to	1	2	3	4
1	0.75	0.1875	0.0625	0
2	0	0.75	0.1875	0.0625
3	0	0	0.75	0.25
4	0	0	0	1

Action: Replace

from \ to	1	2	3	4
1	0.75	0.1875	0.0625	0
2	0.75	0.1875	0.0625	0
3	0.75	0.1875	0.0625	0
4	0.75	0.1875	0.0625	0

Rows are identical, since each month begins with a new machine, regardless of current condition

- Linear Programming Approach
- Policy Iteration Method

Linear Programming Model

What is the policy which minimizes the average cost/month in steady state?

$$\text{Minimize} \sum_{i=1}^{N} C_i^k X_i^k$$

where

X_i^k = probability that machine is in state #i and decision #k is selected

LP Tableau

k:	1	1	2	1	2	2	RHS
i:	1	2	2	3	3	4	
Min	100	200	2100	500	2600	3100	
	0.25	0	¯0.75	0	¯0.75	¯0.75	0
	¯0.1875	0.25	0.8125	0	¯0.1875	¯0.1875	0
	¯0.0625	¯0.1875	¯0.0625	0.25	0.9375	¯0.0625	0
	1	1	1	1	1	1	1

X_i^k = probability that machine is in state #i and decision #k is selected

Iteration 0

Initial policy: keep until the machine fails

basic:	★	★		★		★	
k:	1	1	2	1	2	2	
i:	1	2	2	3	3	4	rhs
Min	0	0	541.463	0	¯304.878	0	¯548.78
	1	0	¯2.04878	0	¯1.17073	0	0.292683
	0	1	1.95122	0	¯1.17073	0	0.292683
	0	0	0.780488	1	2.73171	0	0.317073
	0	0	0.317073	0	0.609756	1	0.097561

i~state, k~action

Initial basis is obtained by selecting one basic variable for each state.

Iteration 0

Steadystate distribution resulting from this policy

Policy: (Cost= 548.78)

State	Action	P{i}
1 Good as new	1 Keep	0.292683
2 Minor deterioration	1 Keep	0.292683
3 Major deterioration	1 Keep	0.317073
4 Failed	2 Replace	0.097561

Choose a column having negative reduced cost, and enter it into the basis:

basic	★	★		★		★	
k:	1	1	2	1	2	2	
i:	1	2	2	3	3	4	rhs
Min	0	0	541.463	0	¯304.878	0	¯548.78
	1	0	¯2.04878	0	¯1.17073	0	0.292683
	0	1	1.95122	0	¯1.17073	0	0.292683
	0	0	0.780488	1	(2.73171)	0	0.317073
	0	0	0.317073	0	0.609756	1	0.097561

choose pivot row, using minimum ratio test

$$\text{minimum} \left\{ \frac{0.31707}{2.7317}, \frac{0.09756}{0.60975} \right\} = \frac{0.31707}{2.7317} = 0.116071$$

Iteration 1

	★	★			★	★	
k:	1	1	2	1	2	2	
i:	1	2	2	3	3	4	rhs
Min	0	0	628.571	111.607	0	0	¯513.393
	1	0	¯1.71429	0.428571	0	0	0.428571
	0	1	2.28571	0.428571	0	0	0.428571
	0	0	0.285714	0.366071	1	0	0.116071
	0	0	0.142857	¯0.223214	0	1	0.0267857

i~state, k~action

X_3^2 has replaced X_3^1 in the basis

Iteration 1

Policy: (Cost= 513.393)

State	Action	P{i}
1 Good as new	1 Keep	0.428571
2 Minor deterioration	1 Keep	0.428571
3 Major deterioration	2 Replace	0.116071
4 Failed	2 Replace	0.0267857

	★	★			★	★	
k:	1	1	2	1	2	2	
i:	1	2	2	3	3	4	rhs
Min	0	0	628.571	111.607	0	0	-513.393
	1	0	-1.71429	0.428571	0	0	0.428571
	0	1	2.28571	0.428571	0	0	0.428571
	0	0	0.285714	0.366071	1	0	0.116071
	0	0	0.142857	-0.223214	0	1	0.0267857

i~state, k~action

Reduced costs are nonnegative...
the optimality condition is satisfied!

Optimal Policy

State	Action
1 Good as new	1 Keep
2 Minor deterioration	1 Keep
3 Major deterioration	2 Replace
4 Failed	2 Replace

Policy Iteration Method

- Average cost per month
- Present value of all future costs

Machine Replacement Example

State	Action
1 Good as new	1 Keep
2 Minor deterioration	1 Keep
3 Major deterioration	1 Keep
4 Failed	2 Replace

g(R) = 548.78

i	Vi
1	-3000
2	-1541.46
3	-195.122
4	0

Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

State #2, Minor deterioration

Current Policy: action #1, Keep

g(R)+Vi(R) = -992.683

k	name	C'	ΔC
1	Keep	-992.683	0
2	Replace	-451.22	541.463

no improvement can be achieved by changing action in this state.

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)

Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

State #3, Major deterioration

Current Policy: action #1, Keep

g(R)+Vi(R) = 353.659

k	name	C'	ΔC
1	Keep	353.659	0
2	Replace	48.7805	-304.878

improve- ment

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)

Machine Replacement Example

State	Action
1 Good as new	1 Keep
2 Minor deterioration	1 Keep
3 Major deterioration	2 Replace
4 Failed	2 Replace

new policy

Value Determination

g(R) = 513.393

i	Vi
1	-3000
2	-1628.57
3	-500
4	0

Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

State #2, Minor deterioration

Current Policy: action #1, Keep

g(R)+Vi(R) = -1115.18

k	name	C'	ΔC
1	Keep	-1115.18	0
2	Replace	-486.607	628.571

no improvement can be achieved by changing action in this state.

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)

Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

State #3, Major deterioration

Current Policy: action #2, Replace

g(R)+Vi(R) = 13.3929

k	name	C'	ΔC
1	Keep	125	111.607
2	Replace	13.3929	0

no improvement can be achieved by changing action in this state.

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)

The current policy is optimal!

Minimizing:
Present value of all future costs
i.e., MDP with discounting is used.

Assume a rate of return of 1.5% per month (18% per year)

$$\beta = \frac{1}{1+r} = \frac{1}{1.015} = 0.985222$$

That is, the present value of a $1 cost next month is $0.985222

Let's begin with an initial policy: keep machine until it fails
i.e., R = (1, 1, 1, 2)

	State	Action
1	Good as new	1 Keep
2	Minor deterioration	1 Keep
3	Major deterioration	1 Keep
4	Failed	2 Replace

Discount factor = 0.985222
(rate of return = 1.5%)

Value Determination

Solve the system of equations:

$$v_i(R) = C_i^{k_i} + \beta \sum_{j\in S} p_{ij}^{k_i} v_j(R) \quad \forall\, i\in S$$

$$P^R = \begin{bmatrix} 0.75 & 0.1875 & 0.0625 & 0 \\ 0 & 0.75 & 0.1875 & 0.0625 \\ 0 & 0 & 0.75 & 0.25 \\ 0.75 & 0.1875 & 0.0625 & 0 \end{bmatrix}$$

$$v_i(R) = C_i^{k_i} + \beta \sum_{j\in S} p_{ij}^{k_i} v_j(R) \quad \forall\, i\in S$$

$$v_i(R) = C_i^{k_i} + \beta \sum_{j\in S} p_{ij}^{k_i} v_j(R) \quad \forall\, i\in S$$

$$\begin{cases} v_1 = 100 + 0.98522\,(0.75v_1 + 0.1875v_2 + 0.0625v_3) \\ v_2 = 200 + 0.98522\,(0.75v_2 + 0.1875v_3 + 0.0625v_4) \\ v_3 = 500 + 0.98522\,(0.75v_3 + 0.25v_4) \\ v_4 = 3100 + 0.98522\,(0.75v_1 + 0.1875v_2 + 0.0625v_3) \end{cases}$$

Solution:

i	V_i
1	35567.4
2	36960.8
3	38299.4
4	38567.4

i	V_i
1	35567.4
2	36960.8
3	38299.4
4	38567.4

That is, $35,567.40 invested at 1.5% per month interest would sufficient to pay all future operation and replacement cost for the machine, if it is initially in state 1, i.e., "good as new"

Policy Improvement

Policy Improvement Step: Evaluation of alternate actions

State #2, Minor deterioration

Current Policy: action #1, Keep

Evaluate alternative action: #2, Replace

$$v_i' = C_i^{k'_i} + \beta \sum_j p_{ij}^{k'_i} v_j \qquad i=2,\ k_i' = 2$$

i	v_i
1	35567.4
2	36960.8
3	38299.4
4	38567.4

$$v_i' = C_i^{k'_i} + \beta \sum_j p_{ij}^{k'_i} v_j \qquad i=2,\ k_i'=2$$

$$v_2' = 2100 + 0.98522\,(0.75v_1 + 0.1875v_2 + 0.675v_3)$$
$$= 2100 + 0.98522\,(0.75\times 35567.4 + 0.1875\times 36960.8 + 0.0625\times 38299.4)$$
$$= 37567.40$$

That is, if we are initially in state 2 and replace the machine, but thereafter follow the original policy R, the present value of all future costs is $27,567.40

Policy Improvement

k	name	V'	ΔV
1	Keep	36960.8	0
2	Replace	37567.4	606.62

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter

ΔV(k) = improvement (if <0)

Since $v_2' > v_2$, the current policy for this state should not be changed.

Policy Improvement

State #3, Major deterioration

Current Policy: action #1, Keep

Evaluate the alternate action: Replace

k	name	V'	ΔV
1	Keep	38299.4	0
2	Replace	38067.4	-232.035

Since $v_3' < v_3$, i.e., $\Delta v_3 = v_3' - v_3 < 0$, the policy in this state should be changed to "replace"

Value Determination

Discount factor = 0.985222
(rate of return = 1.5%)

New policy:

	State	Action	
1	Good as new	1	Keep
2	Minor deterioration	1	Keep
3	Major deterioration	2	Replace
4	Failed	2	Replace

New present values:

i	v_i
1	33799.4
2	35128.6
3	36299.4
4	36799.4

Policy Improvement

State #2, Minor deterioration

Current Policy: action #1, Keep

k	name	V'	ΔV
1	Keep	35128.6	0
2	Replace	35799.4	670.718

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter

ΔV(k) = improvement (if <0)

The policy for this state should not be changed.

Policy Improvement

State #3, Major deterioration

Current Policy: action #2, Replace

k	name	V'	ΔV
1	Keep	36386.1	86.709
2	Replace	36299.4	0

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter

ΔV(k) = improvement (if <0)

The policy for this state should not be changed.

Policy Improvement

No improvement is possible, so the current policy:

	State	Action	
1	Good as new	1	Keep
2	Minor deterioration	1	Keep
3	Major deterioration	2	Replace
4	Failed	2	Replace

is optimal!