Example:

$$\text{Minimize } 2x_1 + 5x_2 + 3x_3$$

subject to

$$x_1 + 3x_2 + x_3 \geq 2$$
$$2x_1 - x_2 + x_3 \geq 2$$
$$x_j \in \{0,1\},\ j = 1,2,3$$

Lagrangian Relaxation of all inequality constraints:

$$L(x_1, x_2, x_3, l_1, l_2) = 2x_1 + 5x_2 + 3x_3$$
$$+ l_1\left[2 - (x_1 + 3x_2 + x_3)\right] + l_2\left[2 - (2x_1 - x_2 + x_3)\right]$$

That is,

$$L(x_1, x_2, x_3, l_1, l_2) =$$
$$(2 - l_1 - 2l_2)x_1 + (5 - 3l_1 + l_2)x_2 + (3 - l_1 - 2l_2)x_3 + 2l_1 + 2l_2$$

We will show that the corresponding Lagrangian dual is equivalent to the LP relaxation!

Lagrangian dual objective:

$$\hat{L}(l_1, l_2) = \underset{x_j \in \{0,1\}}{\text{Minimum}}\ L(x_1, x_2, x_3, l_1, l_2)$$

That is,

$$\hat{L}(l_1, l_2) = 2l_1 + 2l_2 + \underset{x_j \in \{0,1\}}{\text{Minimum}} (2 - l_1 - 2l_2)x_1$$
$$+ (5 - 3l_1 + l_2)x_2 + (3 - l_1 - 2l_2)x_3$$

The solution of the minimization is performed by assigning x_j the value 1 if its coefficient is positive, and 0 otherwise.

Thus, the dual objective function is

$$\hat{L}(l_1, l_2) = 2l_1 + 2l_2 + \min[0, 2 - l_1 - 2l_2]$$
$$+ \min[0, 5 - 3l_1 + l_2] + \min[0, 3 - l_1 - 2l_2]$$

and the *Lagrangian dual* problem is to maximize this objective, i.e.,

$$\underset{l \geq 0}{\text{Maximize}}\ \hat{L}(l_1, l_2)$$

The Lagrangian dual problem is equivalent to the following LP:

$$\text{Maximize } 2l_1 + 2l_2 + q_1 + q_2 + q_3$$

subject to

$$q_1 \le 0 \ \& \ q_1 \le 2 - l_1 - 2l_2$$
$$q_2 \le 0 \ \& \ q_2 \le 5 - 3l_1 + l_2$$
$$q_3 \le 0 \ \& q_3 \le 3 - l_1 - 2l_2$$
$$l_1 \ge 0,\ l_2 \ge 0$$

Make the change of variable $\varphi_i = -\theta_i$:

$$\text{Maximize } 2l_1 + 2l_2 - j_1 - j_2 - j_3$$
subject to
$$l_1 + 2l_2 - j_1 \le 2$$
$$3l_1 - l_2 - j_2 \le 5$$
$$l_1 + 2l_2 - j_3 \le 3$$
$$l_i \ge 0,\ i = 1,2;\ j_j \ge 0,\ j = 1,2,3$$

The ordinary LP dual of this problem is:

$$\text{Minimize } 2x_1 + 5x_2 + 3x_3$$
subject to
$$x_1 + 3x_2 + x_3 \ge 2$$
$$2x_1 - x_2 + 2x_3 \ge 2$$
$$-x_1 \ge -1, \text{ i.e., } x_1 \le 1$$
$$-x_2 \ge -1, \text{ i.e., } x_2 \le 1$$
$$-x_3 \ge -1, \text{ i.e., } x_3 \le 1$$
$$x_j \ge 0,\ j=1,2,3$$

That is,

$$\text{Minimize } 2x_1 + 5x_2 + 3x_3$$
subject to
$$x_1 + 3x_2 + x_3 \ge 2$$
$$2x_1 - x_2 + 2x_3 \ge 2$$
$$0 \le x_j \le 1,\ j=1,2,3$$

This is the *LP relaxation* of the original ILP!

