A Puzzle: Three Jealous Husbands

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The following is typical of a common class of puzzles:

Three couples (husbands & wives) must get to town via a Corvette with a capacity of only two persons.

How might they do this, taking several trips, so that no wife is ever left at either source or destination with either of the other women's husbands unless her own husband is also present?

To analyze this problem, define $2^6 = 64$ possible *states* of the "system", each denoted by a binary vector X of length 6, where

$$X_i = \begin{cases} 1 & \text{if individual \#i is at the destination} \\ 0 & \text{if individual \#i is at the origin} \end{cases}$$

i	individual
1	Husband #1
2	Wife #1
3	Husband #2
4	Wife #2
5	Husband #3
6	Wife #3

For example, the system begins in state (0,0,0,0,0,0) and should end in state (1,1,1,1,1,1)

#	State
goal → 1	1 1 1 1 1 1
2	1 1 1 1 1 0
3	1 1 1 1 0 0
4	1 1 1 0 1 1
5	1 1 1 0 1 0
6	1 1 0 0 1 1
7	1 1 0 0 0 0
8	1 0 1 1 1 1
9	1 0 1 1 1 0
10	1 0 1 0 1 1
11	1 0 1 0 1 0

#	State
12	0 1 0 1 0 1
13	0 1 0 1 0 0
14	0 1 0 0 0 1
15	0 1 0 0 0 0
16	0 0 1 1 1 1
17	0 0 1 1 0 0
18	0 0 0 1 0 1
19	0 0 0 1 0 0
20	0 0 0 0 1 1
21	0 0 0 0 0 1
initial state → 22	0 0 0 0 0 0

Not all of the 64 states are feasible, e.g., in the state (1,0,0,1,0,0) wife #1 is at the origin, together with both husbands #2 & 3, while her husband is at the destination!

42 of the states are infeasible in a similar way, leaving only 22 feasible states.

With each trip of the Corvette, the "system" changes states, i.e., makes a *transition* .

For example, if Husband #1 and Wife #1 leave together initially, then the system makes the transition

(0,0,0,0,0,0) —*depart*→ (1,1,0,0,0,0)
State #22 *State #7*

Likewise, an *arrival* of Husband #1 and Wife #1 would result in a transition:

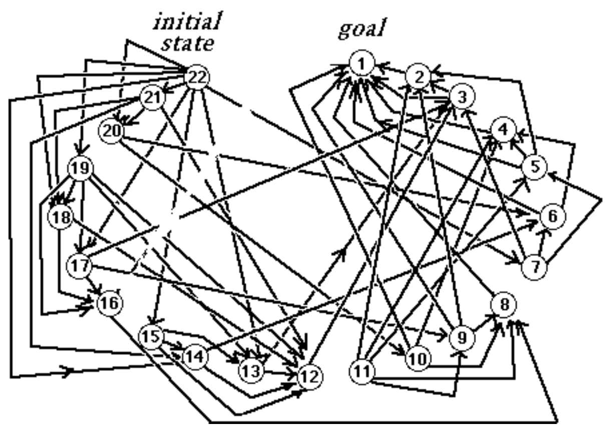
(0,0,0,0,0,0) ←*arrive*— (1,1,0,0,0,0)
State #22 *State #7*

We seek a sequence of transitions starting at state #22 and ending at state #1, with the property that the sequence begins and ends with a *departure* from the origin, with alternate transtions corresponding to *arrivals* at the origin.

Departure transitions

A =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	1	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
16	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	0	0
20	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	1	0	0
22	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1	1	1	0



Departure transitions

The arcs in this digraph represent *departures* from the origin. The digraph representing *arrivals* at the origin would be identical, *except that* the directions of the arcs are reversed!

The *transpose* of the "departure" transition matrix A gives the "arrival" transition matrix!

Arrival transitions

$A^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
2	0	0	1	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0
4	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
8	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	1	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	0	1	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

or *and*

The generalized inner product "∨.∧" of the departure and the arrival transition matrices therefore indicates the transitions resulting from a round trip of the Corvette:

Round-Trip Transitions

$A \vee.\wedge A^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
3	0	1	1	1	1	1	0	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0
4	0	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
5	0	1	1	1	1	1	0	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0
6	0	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	1	0	1	1	0	0	1	0	0	1	0	0
8	0	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
9	0	1	1	1	1	1	0	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0
10	0	1	1	1	1	1	0	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0
11	0	0	1	0	1	0	1	0	1	1	1	0	0	0	0	1	1	0	0	1	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1	0	1	0
14	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	0	1	1	1	1	0
15	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	0	1	1
16	0	1	1	1	1	1	0	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0
17	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	1	0	1	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1	1	0
19	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	0	1	1
20	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0	1	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	0	1	1
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	1

The transition matrix corresponding to a sequence of *departure-arrival-departure* could be computed by an ∨.∧ inner product of the sequence of corresponding transition matrices:

$$A \vee.\wedge A^T \vee.\wedge A$$

The path that we are seeking corresponds to an entry in the matrix

$$A \vee.\wedge A^T \vee.\wedge A \vee.\wedge A^T \cdots \vee.\wedge A$$

depart *arrive* *depart* *arrive* *depart*

One-and-a-half Trip Transitions

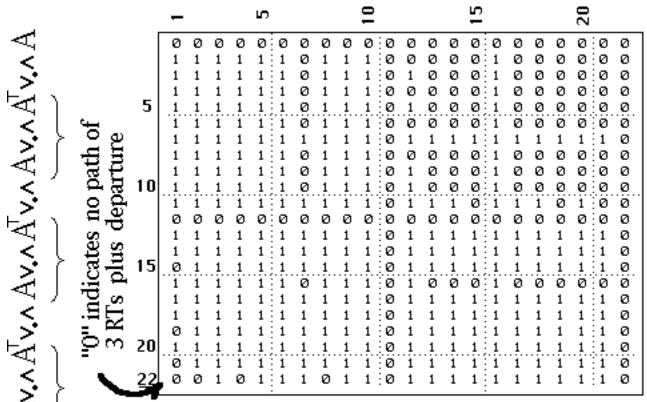
	1				5					10					15					20		
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	1	1	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0
	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
10	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	1	0	1	1	1	0	1	0	0	1	1	1	0	1	1	1	0	1	0	0
	0	0	1	0	1	1	1	0	0	1	0	1	1	1	0	1	1	1	0	1	0	0
15	0	0	1	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0
	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	1	1	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0
	0	0	1	0	0	1	1	0	1	1	0	1	1	1	0	1	1	1	0	1	0	0
	0	0	1	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0
20	0	1	1	1	1	1	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0
	0	0	1	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0
22	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	0	1	1	1	1	1	0

$A \vee.\wedge A^T \vee.\wedge A$

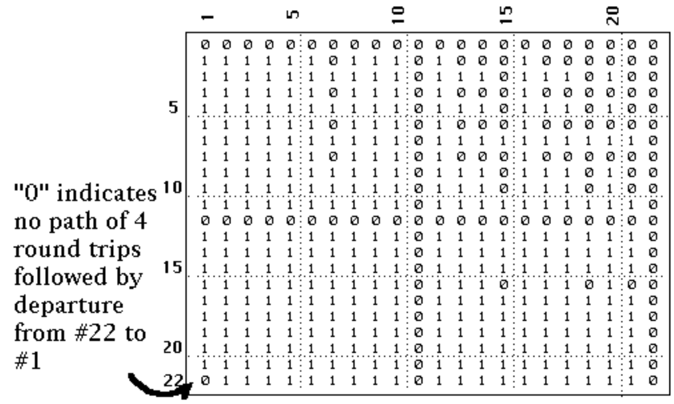
$A \vee.\wedge A^T \vee.\wedge A \vee.\wedge A^T \vee.\wedge A$

	1				5					10					15					20		
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1	0	1	0	0
	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
10	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
	1	1	1	1	1	1	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0
	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0
15	0	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	0
	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1	0	1	0	0
	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0
	0	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	0
20	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1	0	1	0	0
	0	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	0
22	0	0	1	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0

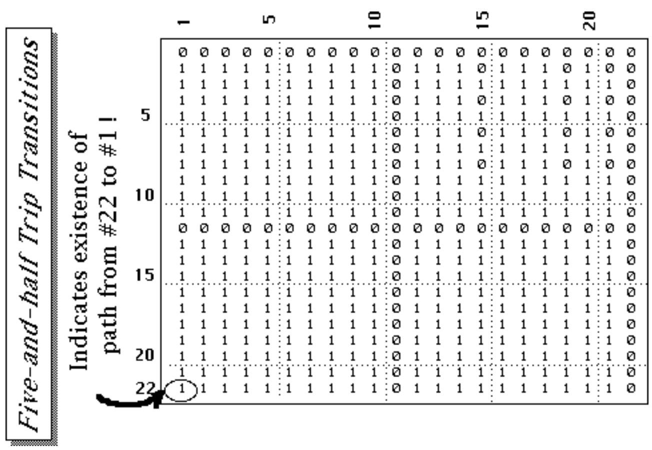
Two Round Trips, followed by departure



Three round trips, followed by departure



Four round trips, followed by departure



Hence, there exists (at least one) path of the type we desire (departure-arrival pairs, followed by a departure) from node #22 to node #1,

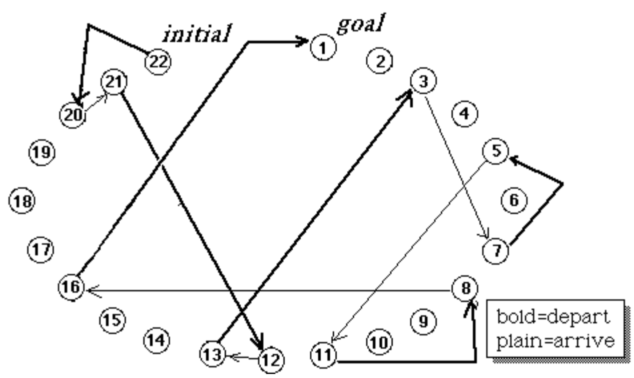
which consists of

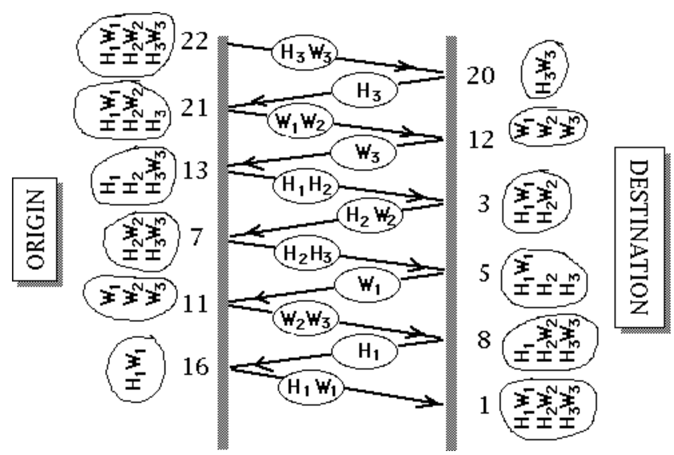
5 round trips and a departure.

Indentifying the arcs (transitions) along this path requires an examination of the v.∧ computations which result in "1".

One Solution:

22 - 20 - 21 - 12 - 13 - 3 - 7 - 5 - 11 - 8 - 16 - 1





Obviously, by permuting the indices of the couples, we obtain essentially the same solution! (E.g., relabel the couples by the indices 3,1,2 instead of 1,2,3.)

Are all of the solutions obtainable by permuting the indices of the previous solution?

Cf. the book "Introductory Graph Theory", by Gary Chartrand, Dover Bks, § 6.4