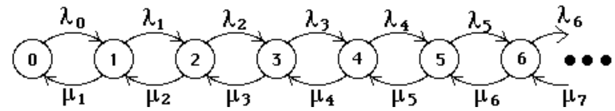
Birth-Death Processes

Birth-Death Process

A birth-death process is a continuous-time Markov chain which models the size of a population; the population increases by 1 ("birth") or decreases by 1 ("death").

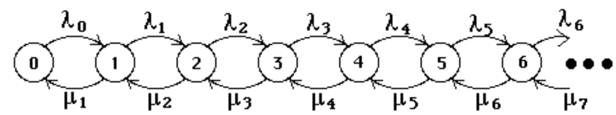


Steadystate Probabilities

To calculate steady-state probability distribution, we use "Balance" equations:

Rate at which system enters state #i
= Rate at which system leaves state #i

Steady-State Distribution of a Birth-Death Process

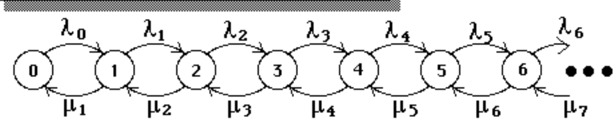


Balance Equations:

State 0: $\lambda_0\pi_0 = \mu_1\pi_1 \Rightarrow \boxed{\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0}$

Rate leaving state #0 = Rate entering state #0

Steady-State Distribution of a Birth-Death Process



Balance Equations:

State 1: *Rate leaving state #1 = Rate entering state #1*

$$(\lambda_1 + \mu_1)\,\pi_1 = \lambda_0\pi_0 + \mu_2\pi_2$$

$$\Rightarrow \pi_2 = \frac{(\lambda_1 + \mu_1)\,\pi_1 - \lambda_0\pi_0}{\mu_2} = \frac{(\lambda_1 + \mu_1)\frac{\lambda_0}{\mu_1}\pi_0 - \lambda_0\pi_0}{\mu_2}$$

$$\Rightarrow \boxed{\pi_2 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0}$$

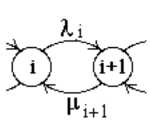
In general,

$$(\lambda_{i-1} + \mu_{i-1})\,\pi_{i-1} = \lambda_{i-2}\pi_{i-2} + \mu_i\pi_i$$

$$\Rightarrow \boxed{\pi_i = \frac{\lambda_{i-1}\cdots\lambda_1\lambda_0}{\mu_i\cdots\mu_2\mu_1}\,\pi_0} \quad i=1,2,3,\ldots$$

$$\pi_i = \left(\frac{\lambda_{i-1}}{\mu_i}\right)\cdots\left(\frac{\lambda_1}{\mu_2}\right)\left(\frac{\lambda_0}{\mu_1}\right)\pi_0$$

$$= \rho_{i-1}\cdots\rho_1\,\rho_0\,\pi_0 \quad \text{where } \rho_i = \frac{\lambda_i}{\mu_{i+1}}$$



ratio of transition rates between adjacent states

Substituting these expressions for π_i into $\sum_{i=0}^{\infty}\pi_i = 1$ yields:

$$\pi_0 + \sum_{i=1}^{\infty}\frac{\lambda_{i-1}\cdots\lambda_1\lambda_0}{\mu_i\cdots\mu_2\mu_1}\,\pi_0 = 1$$

$$\Rightarrow \pi_0\left[1 + \sum_{i=1}^{\infty}\frac{\lambda_{i-1}\cdots\lambda_1\lambda_0}{\mu_i\cdots\mu_2\mu_1}\right] = 1$$

$$\Rightarrow \boxed{\frac{1}{\pi_0} = \left(1 + \sum_{i=1}^{\infty}\frac{\lambda_{i-1}\cdots\lambda_1\lambda_0}{\mu_i\cdots\mu_2\mu_1}\right)}$$

Once π_0 is evaluated by computing the reciprocal of this infinite sum, π_i is easily computed for each i=1, 2, 3, ...

$$\boxed{\frac{1}{\pi_0} = \left[1 + \sum_{i=1}^{\infty} \frac{\lambda_{i-1}\cdots\lambda_1\lambda_0}{\mu_i\cdots\mu_2\mu_1}\right]}$$

$$\boxed{\pi_i = \frac{\lambda_{i-1}\cdots\lambda_1\lambda_0}{\mu_i\cdots\mu_2\mu_1}\,\pi_0} \quad i=1,2,3, \ldots$$

Examples

- ☞ Backup Computer System
- ☞ Gasoline Station
- ☞ Ticket Sales by Phone

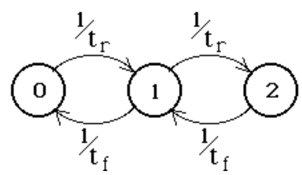
Example

An airlines reservation system has 2 computers, one on-line and one standby. The operating computer fails after an exponentially-distributed duration having mean t_f and is then replaced by the standby computer.

There is one repair facility, and repair times are exponentially-distributed with mean t_r.

What fraction of the time will the system fail, i.e., both computers having failed?

Let X(t) = number of computers in operating condition at time t. Then X(t) is a birth-death process.



Note that the birth rate in state 2 is zero!

$$\frac{1}{\pi_0} = 1 + \frac{1/t_r}{1/t_f} + \left(\frac{1/t_r}{1/t_f}\right)^2$$

$$\frac{1}{\pi_0} = 1 + \frac{t_f}{t_r} + \left(\frac{t_f}{t_r}\right)^2$$

$$\pi_0 = \frac{t_r^2}{t_r^2 + t_r t_f + t_f^2}$$

probability that both computers have failed

Suppose that $\frac{t_f}{t_r} = 10$, i.e., the average repair time is 10% of the average time between failures:

$$\frac{1}{\pi_0} = 1 + 10 + 100 = 111$$

$$\pi_0 = \frac{1}{111} = 0.009009$$

Then both computers will be simultaneously out of service 0.9% of the time.

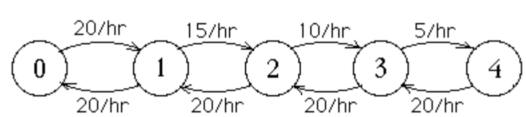
A gasoline station has only one pump.
Cars arrive at the rate of 20/hour.
However, if the pump is already in use, these potential customers may "balk", i.e., drive on to another gasoline station.

If there are n cars already at the station, the probability that an arriving car will balk is $n/4$, for n=1,2,3,4 , and 1 for n>4.

Time required to service a car is exponentially distributed, with mean = 3 minutes.

What is the expected waiting time of customers?

"Birth/death" model:



$$\frac{1}{\pi_0} = 1 + \frac{20}{20} + \frac{20}{20}\times\frac{15}{20} + \frac{20}{20}\times\frac{15}{20}\times\frac{10}{20} + \frac{20}{20}\times\frac{15}{20}\times\frac{10}{20}\times\frac{5}{20}$$

$$= 1 + 1 + 0.75 + 0.375 + 0.09375 = 3.21875$$

$$\pi_0 = 0.3106796$$

Steady State Distribution

$\pi_0 = 0.3106796,$

$\pi_1 = \pi_0 = 0.3106796,$

$\pi_2 = 0.75\pi_0 = 0.2330097,$

$\pi_3 = 0.375\pi_0 = 0.1165048,$

$\pi_4 = 0.09375\pi_0 = 0.0291262$

Average Number in System

$$L = \sum_{i=0}^{4} i\,\pi_i$$

$$= 0.3106796 + 2(0.2330097) + 3(0.1165048) + 4(0.0291262)$$

$$= 1.2427183$$

Average Arrival Rate

$$\bar{\lambda} = \sum_{i=0}^{4} \lambda_i\,\pi_i$$

$$= (0.3106796)\times 20/\text{hr} + (0.3106796)\times 15/\text{hr} + (0.2330097)\times 10/\text{hr} + (0.1165048)\times 5/\text{hr} + (0.0291262)\times 0/\text{hr}$$

$$= 13.786407/\text{hr}$$

Average Time in System

$$W = L/\bar{\lambda} = \frac{1.2427183}{13.786407/\text{hr}}$$

$$= 0.0901408 \text{ hr.} = 5.40844504 \text{ minutes}$$

Hancher Auditorium has 2 ticket sellers who answer phone calls & take incoming ticket reservations, using a single phone number.

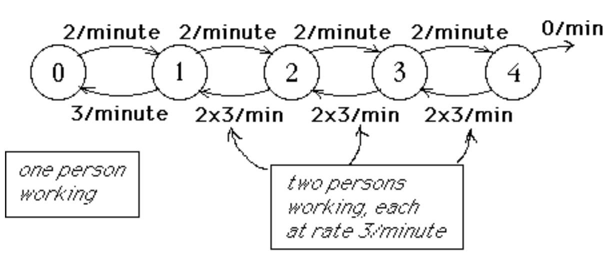
In addition, 2 callers can be put "on hold" until one of the two ticket sellers is available to take the call.

If all 4 phone lines are busy, a caller will get a busy signal, and waits until later before trying again.

Calls arrive at an average rate of 2/minute, and ticket reservations service time averages 20 sec. and is exponentially distributed.

What is...

- the fraction of the time that each ticket seller is idle?
- the fraction of customers who get a busy signal?
- the average waiting time ("on hold")?



BIRTH-DEATH MODEL