# Gradient Projection Algorithm

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Consider the problem of optimizing a nonlinear function subject to linear constraints:

$$\text{Minimize } f(x) \quad \text{subject to} \quad Ax \le b', \quad Ex = b''$$



Suppose that

$$Mx^o = b$$

and that we wish to choose a step direction d so that

$$M(x^o + d) = b$$

Then

$$Mx^o + Md = b$$
$$b + Md = b$$
$$Md = 0$$

i.e., d lies in the null space of the matrix M:

$$\{\, x \mid Mx = 0 \,\}$$



Assume that M is mxn with full row rank, i.e., rank M = m, so that there are no linearly dependent rows.

Then $MM^T$ is mxm with rank m, so that $MM^T$ is nonsingular, i.e., $(MM^T)^{-1}$ exists.

Consider Py, where y is an arbitrary vector and

$$P = I - M^T(MM^T)^{-1}M$$



Consider the matrix P defined by

$$P = I - M^T(MM^T)^{-1}M$$

for arbitrary matrix M with full row rank.

This is a *projection* matrix if

- $P = P^T$ *i.e., P is symmetric*
- $PP = P$ *i.e., the projection of the projection of a vector is the same as the projection!*



*To show:* • $P = P^T$

$$P^T = \left[I - M^T(MM^T)^{-1}M\right]^T$$
$$= I - \left[M^T(MM^T)^{-1}M\right]^T$$
$$= I - M^T\left((MM^T)^{-1}\right)^T(M^T)^T$$
$$= I - M^T\left[(M^T)^T M^T\right]^{-1}M$$
$$= I - M^T(MM^T)^{-1}M = P$$



*To show:* • $PP = P$

$$PP = \left(I - M^T(MM^T)^{-1}M\right)\left(I - M^T(MM^T)^{-1}M\right)$$
$$= I - 2M^T(MM^T)^{-1}M + M^T(MM^T)^{-1}\underbrace{MM^T(MM^T)^{-1}}_{I}M$$
$$= I - 2M^T(MM^T)^{-1}M + M^T(MM^T)^{-1}\, I\, M$$
$$= I - M^T(MM^T)^{-1}M = P$$



What is the result of multiplying the projection Py by M?

$$M(Py) = M\left(I - M^T(MM^T)^{-1}M\right)y$$
$$= My - MM^T(MM^T)^{-1}My$$
$$= My - I\,My = 0$$

That is, P projects y onto the null space of the matrix M.

Given an initial feasible solution $X^o$, Rosen's Gradient Projection Algorithm projects the steepest descent direction onto the null space of the tight constraints, so that the resulting direction will be feasible.

## Rosen's Gradient Projection Algorithm

**Step 0** Choose an initial feasible point $x^o$, and let $Mx^o = b$ be the tight constraints. Let k=0.

**Step 1** If M is vacuous (# rows =0), let P=I; otherwise, let

$$P = I - M^T\left(MM^T\right)^{-1}M$$



**Step 2** Let $d = -P\,\nabla f(x^k)$ be the search direction.

If $d^k = 0$, then let $\lambda = -\left(MM^T\right)^{-1}M\,\nabla f(x^k)$

*LAGRANGE MULTIPLIERS*

If $\lambda_i \geq 0$ for each inequality $M_i x \leq b_i$
STOP; $x^k$ is optimal.
Otherwise, if $\lambda_i < 0$ for some inequality $M_i x \leq b_i$
remove row i from the matrix M of binding constraints, and return to step 1.



**Step 3** Perform a one-dimensional search to

Minimize $f(x^k + t\,d^k)$
subject to $0 \leq t \leq t_{max}$

where

$$t_{max} = \min \left\{ \hat{b}_j / \hat{d}_j \mid \hat{b}_j > 0,\ \hat{d}_j > 0 \right\}$$

$$\hat{b} = b - Ax^k,\quad \hat{d} = Ad^k$$

Let $x^{k+1} = x^k + t^* d^k$, increment k, let $Mx^k = b$ be the new binding constraints, and go to step 1.



Suppose that $d = P\left(-\nabla f(x)\right) = 0$
i.e.,

$$-\left(I - M^T\left(MM^T\right)^{-1}M\right)\nabla f(x) = 0$$

$$\Longrightarrow\quad -\nabla f(x) + M^T\left(MM^T\right)^{-1}M\nabla f(x) = 0$$

$$\Longrightarrow\quad -\nabla f(x) - M^T\lambda = 0$$

$$\Longrightarrow\quad -\nabla f(x) = \sum_i M_i\lambda_i$$

where $\lambda = -\left(MM^T\right)^{-1}M\,\nabla f(x^k)$
and $M_i$ = row #i of matrix M

*= gradient of tight constraint # i*

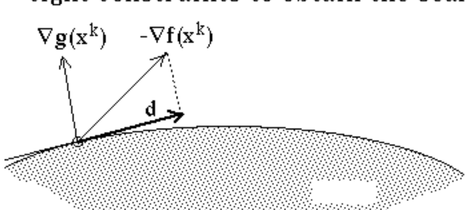


Then $x^k$ & $\lambda$ satisfy the K-K-T conditions if $\lambda_i \geq 0$ for each inequality $M_i x \leq b_i$

If $\lambda_i < 0$ for inequality $M_i x \leq b_i$, then the steepest descent direction $-\nabla f(x)$ points *into* the feasible region of this constraint, so that the constraint should be made loose, i.e., non-binding.



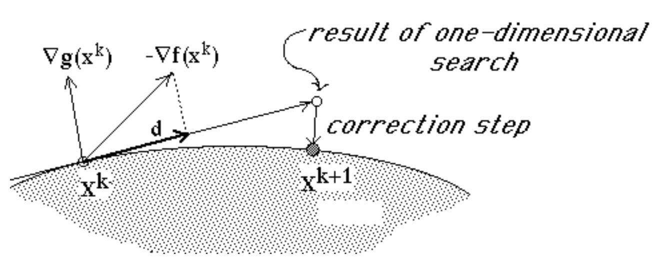
**Nonlinear Constraints** If the constraints are nonlinear, then the steepest descent direction should be projected onto the null space of the gradients of the tight constraints to obtain the search direction.




$$d = -P\nabla f(x^k)$$



After the one-dimensional search in this direction, a "correction" step will generally be required to regain feasibility *(e.g., using Newton-Raphson method.)*



**EXAMPLE**

Minimize $2x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1 - 6x_2$

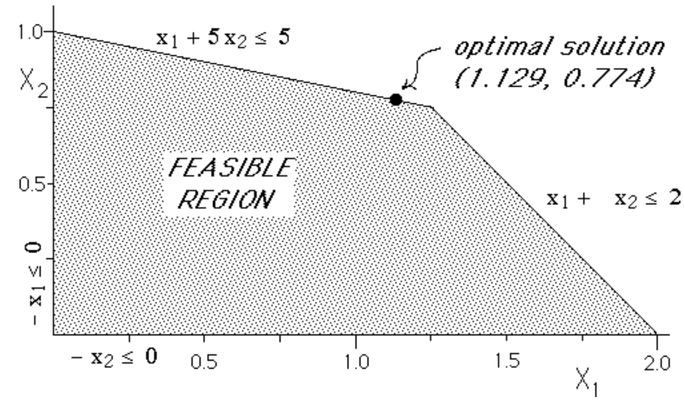
**subject to**

$$x_1 + x_2 \le 2$$
$$x_1 + 5x_2 \le 5$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

*NONNEGATIVITY CONSTRAINTS*

*Actually, this is a QP problem, and would be better solved by a QP algorithm!*








Objective

```
Z←F X;Q
⍝
⍝        Sample NLP problem for
⍝      Gradient Projection Algorithm
⍝      (Quadratic objective function)
⍝
Q←2 2⍴2 ¯1 ¯1 2
Z←(X+.×Q+.×X)+(¯4 ¯6+.×X)
```

Gradient of objective

```
G←GRADIENT X;Q
⍝
⍝     Gradient for objective function
⍝          of Sample NLP problem
⍝
Q←2 2⍴2 ¯1 ¯1 2
G←(2×Q+.×X)+(¯4 ¯6)
```



Inequality Constraints

```
 1  1 ≤  2
 1  5 ≤  5
¯1  0 ≤  0
 0 ¯1 ≤  0
```

Equality Constraints

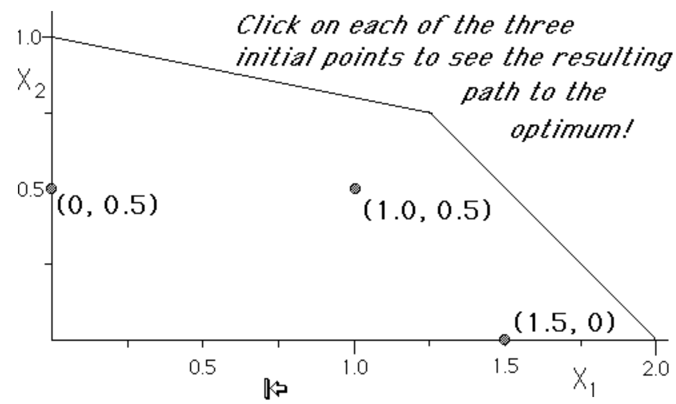
——None——



Gradient Projection Algorithm

```
Tolerance for search direction = 0.001
(TOL ≥ +/|Projection of ∇f(x) onto tight constraints)
Maximum # of Gradient Projection iterations = 25
Tolerance for one-dimensional searches = 0.005
Maximum # of 1-dimensional search iterations = 25
```



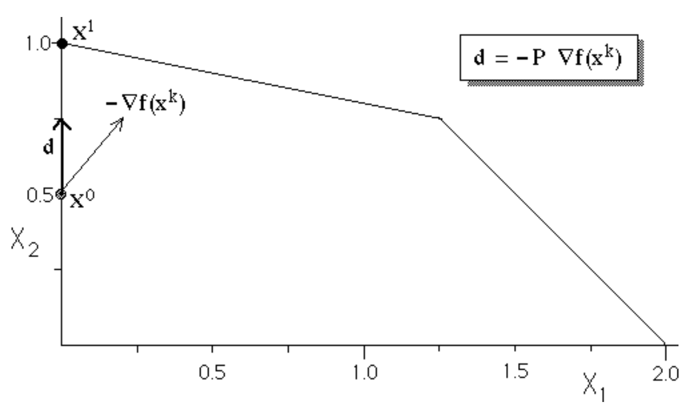





Iteration 1

```
X= 0 0.5
F(X) = ¯2.5
Slack variables: 1.5 2.5 0 0.5

Constraint Partition: Tight: 3          Slack: 1 2 4
Matrix M =
                               ¯1   0

Projection Matrix P =
                               0 0
                               0 1

Gradient ∇f(x) = ¯5 ¯4

Search Direction = 0 4
Maximum step size = 0.125
Optimal step size = 0.125
```



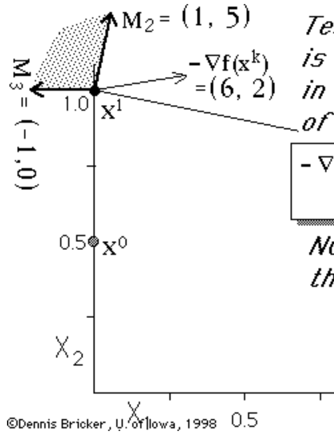


Iteration 2

```
X= 0 1
F(X) = ¯4
Slack variables: 1 0 0 1

Constraint Partition: Tight: 2 3          Slack: 1 4
Matrix M =
                    [ 1  5 ]
                    [-1  0 ]

Projection Matrix P =
                    [ 0 0 ]
                    [ 0 0 ]

Gradient ∇f(x) = ¯6 ¯2
Search Direction = 0 0
```

$= d$






*Test K-K-T conditions.... is steepest descent direction in the cone of the gradients of the tight constraints?*

$$-\nabla f(x^1) = \lambda_2 M_2 + \lambda_3 M_3$$
$$\lambda_2 \geq 0, \lambda_3 \geq 0$$

*NO! Therefore, "release" the offending constraint!*

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} = 0.4 \begin{bmatrix} 1 \\ 5 \end{bmatrix} - 5.6 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



*Test K-K-T conditions....*

$$-\nabla f(x^1) = \lambda_2 M_2 + \lambda_3 M_3$$
$$\lambda_2 \geq 0, \lambda_3 \geq 0$$

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} = 0.4 \begin{bmatrix} 1 \\ 5 \end{bmatrix} - 5.6 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

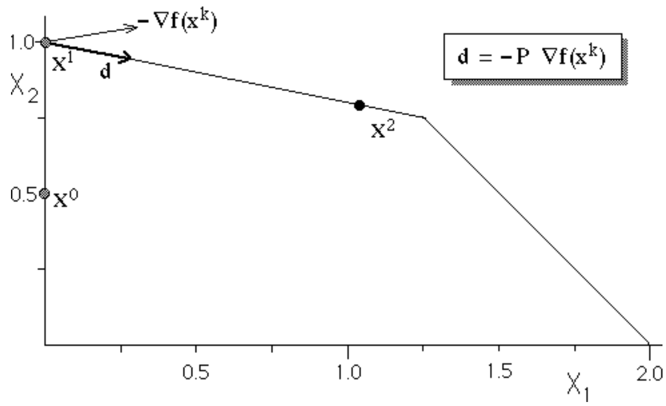
*negative!*

```
Lagrange Multipliers = 0.4 ¯5.6
***Release Tight Constraint 2
Constraint Partition: Tight: 2          Slack: 1 4 3
```



```
X= 0 1
F(X) = ¯4
Slack variables: 1 0 0 1

Constraint Partition: Tight: 2          Slack: 1 4 3
Matrix M =
                    [1  5]

Projection Matrix P =
                    [ 0.961538 ¯0.192308 ]
                    [¯0.192308  0.0384615]

Gradient ∇f(x) = ¯6 ¯2
Search Direction = 5.38462 ¯1.07692

Maximum step size = 0.232143
Optimal step size = 0.209677
```





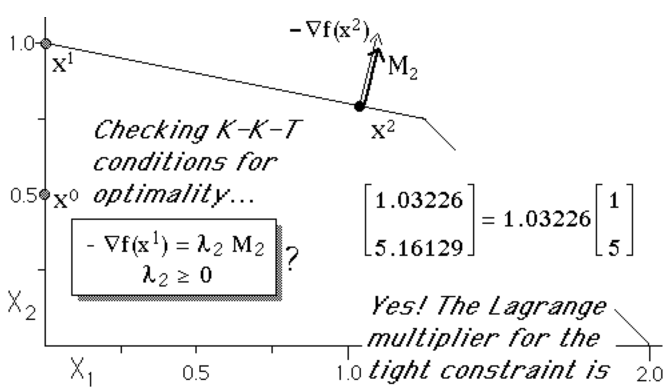

$$d = -P\ \nabla f(x^k)$$

Iteration 3

```
X= 1.12903 0.774194
F(X) = ¯7.16129
Slack variables: 0.0967742 0 1.12903 0.774194

Constraint Partition: Tight: 2          Slack: 1 3 4
Matrix M =
                    [1  5]

Projection Matrix P =
                    [ 0.961538 ¯0.192308 ]
                    [¯0.192308  0.0384615]

Gradient ∇f(x) = ¯1.03226 ¯5.16129

Search Direction = 3.33067E¯16 ¯2.22045E¯16
```

$= d \approx 0$






*Checking K-K-T conditions for optimality...*

$$-\nabla f(x^1) = \lambda_2 M_2 \quad ?$$
$$\lambda_2 \geq 0$$

$$\begin{bmatrix} 1.03226 \\ 5.16129 \end{bmatrix} = 1.03226 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

*Yes! The Lagrange multiplier for the tight constraint is positive!*

```
Lagrange Multipliers = 1.03226

***Optimality Conditions Satisfied
     X= 1.12903 0.774194
     F(X) = ¯7.16129

Lagrange Multipliers= 1.03226
Slack in inequality constraints: 0.0967742 0 1.12903 0.774
```

Iteration 1

X= 1 0.5
F(X) = ¯5.5
Slack variables: 0.5 1.5 1 0.5

Constraint Partition: Tight: none Slack: 1 2 3 4
Matrix M =

Gradient ∇f(x) = ¯1 ¯6
Search Direction = 1 6

Maximum step size = 0.0483871
Optimal step size = 0.0483871



Iteration 2

X= 1.04839 0.790323
F(X) = ¯7.14516
Slack variables: 0.16129 0 1.04839 0.790323

Constraint Partition: Tight: 2 Slack: 1 3 4
Matrix M =
$$\begin{bmatrix} 1 & 5 \end{bmatrix}$$

Projection Matrix P =
$$\begin{bmatrix} 0.961538 & ^-0.192308 \\ ^-0.192308 & 0.0384615 \end{bmatrix}$$

Gradient ∇f(x) = ¯1.3871 ¯4.93548
Search Direction = 0.384615 ¯0.0769231

Maximum step size = 0.524194
Optimal step size = 0.209677



Iteration 3

X= 1.12903 0.774194
F(X) = ¯7.16129
Slack variables: 0.0967742 0 1.12903 0.774194

Constraint Partition: Tight: 2 Slack: 1 3 4
Matrix M =
$$\begin{bmatrix} 1 & 5 \end{bmatrix}$$

Projection Matrix P =
$$\begin{bmatrix} 0.961538 & ^-0.192308 \\ ^-0.192308 & 0.0384615 \end{bmatrix}$$

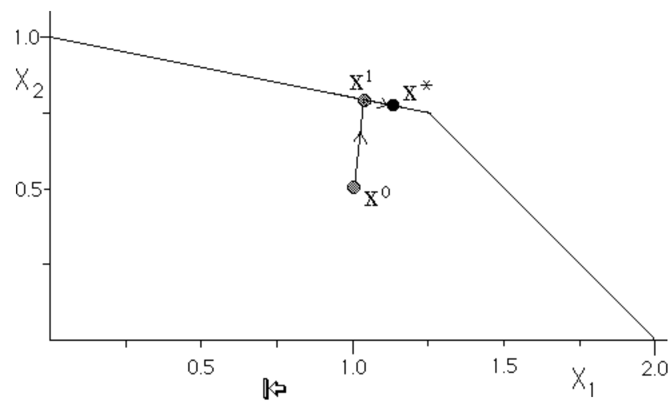
Gradient ∇f(x) = ¯1.03226 ¯5.16129
Search Direction = 3.33067E¯16 ¯2.22045E¯16 ≈ 0
Lagrange Multipliers = 1.03226

***Optimality Conditions Satisfied
X= 1.12903 0.774194
F(X) = ¯7.16129

Lagrange Multipliers= 1.03226

Slack in inequality constraints: 0.0967742 0 1.12903 0.7741






Iteration 1

X= 1.5 0
F(X) = ¯1.5
Slack variables: 0.5 3.5 1.5 0

Constraint Partition: Tight: 4 Slack: 1 2 3
Matrix M =
$$\begin{bmatrix} 0 & ^-1 \end{bmatrix}$$

Projection Matrix P =
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Gradient ∇f(x) = 2 ¯9
Search Direction = ¯2 0

Maximum step size = 0.75
Optimal step size = 0.25



Iteration 2

X= 1 0
F(X) = ¯2
Slack variables: 1 4 1 0

Constraint Partition: Tight: 4 Slack: 1 2 3
Matrix M =
$$\begin{bmatrix} 0 & ^-1 \end{bmatrix}$$

Projection Matrix P =
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Gradient ∇f(x) = 0 ¯8
Search Direction = 0 0

Lagrange Multipliers = ¯8

***Release Tight Constraint 1
Constraint Partition: Tight: none Slack: 1 2 3 4



X= 1 0
F(X) = ¯2
Slack variables: 1 4 1 0

Constraint Partition: Tight: none Slack: 1 2 3 4
Matrix M =

Gradient ∇f(x) = 0 ¯8
Search Direction = 0 8

Maximum step size = 0.1
Optimal step size = 0.1

Iteration 3

X= 1 0.8
F(X) = ¯7.12
Slack variables: 0.2 0 1 0.8
Constraint Partition: Tight: 2 Slack: 1 3 4
Matrix M =
1 5

Projection Matrix P =

| | |
|---|---|
| 0.961538 | ¯0.192308 |
| ¯0.192308 | 0.0384615 |

Gradient ∇f(x) = ¯1.6 ¯4.8
Search Direction = 0.615385 ¯0.123077
Maximum step size = 0.40625
Optimal step size = 0.209677



Iteration 4

X= 1.12903 0.774194
F(X) = ¯7.16129
Slack variables: 0.0967742 0 1.12903 0.774194
Constraint Partition: Tight: 2 Slack: 1 3 4
Matrix M =
1 5

Projection Matrix P =

| | |
|---|---|
| 0.961538 | ¯0.192308 |
| ¯0.192308 | 0.0384615 |

Gradient ∇f(x) = ¯1.03226 ¯5.16129
Search Direction = 4.77396E¯15 ¯1.08247E¯15 ≈ 0
Lagrange Multipliers = 1.03226

***Optimality Conditions Satisfied
X= 1.12903 0.774194
F(X) = ¯7.16129

Lagrange Multipliers= 1.03226
Slack in inequality constraints: 0.0967742 0 1.12903 0.774