Path-Following Algorithm for Posynomial Geometric Programming

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Consider the linearly-constrained convex programming problem

$$\Phi^* = \text{Minimum } F(x) \quad \text{subject to } Ax=b, \; x \geq 0$$

where F is a convex function.

Barrier function:

$$\Phi(\zeta) = \text{Minimum } F(x) - \zeta \sum_{j=1}^{n} \ln x_j \quad \text{subject to } Ax=b, \; x \geq 0$$

as $x \longrightarrow 0$, $-\zeta \ln(x) \longrightarrow \infty$

so the minimizer $x(\zeta)$ of $\Phi(\zeta)$ is positive!

Lagrangian function:

$$L(x,y,\zeta) = F(x) - \zeta \sum_{j=1}^{n} x_j - y^T (Ax - b)$$

Optimality conditions for $\Phi(\zeta)$:

$$\frac{\partial L(x,y,\zeta)}{\partial x_j} \geq 0, \quad j=1,\ldots n$$

$$\frac{\partial L(x,y,\zeta)}{\partial y_i} = 0, \quad i=1,\ldots m$$

$$x_j \left(\frac{\partial L(x,y,\zeta)}{\partial x_j} \right) = 0$$

complementary slackness conditions

$$\frac{\partial L(x,y,\zeta)}{\partial x_j} \geq 0, \quad j=1,\ldots n$$

$$\Rightarrow \quad \frac{\partial}{\partial x_j} L(x,y,\zeta) = \frac{\partial F}{\partial x_j} - \frac{\zeta}{x_j} - \sum_{i=1}^{m} y_i a_{ij} \geq 0$$

$$\frac{\partial L(x,y,\zeta)}{\partial y_i} = 0, \quad i=1,\ldots m$$

$$\Rightarrow \quad Ax = b$$

Notation

$$e^T \equiv (1, 1, 1, \ldots 1)$$

$$X \equiv \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & x_n \end{bmatrix} \Rightarrow X^{-1} = \begin{bmatrix} \frac{1}{x_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{x_2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & \frac{1}{x_n} \end{bmatrix}$$

$$\frac{\partial L(x,y,\zeta)}{\partial x_j} \geq 0, \quad j=1,\ldots n$$

$$\Rightarrow \quad \nabla F(x) - A^T y - \zeta X^{-1} e \geq 0$$

$$x \left(\nabla F(x) - A^T y - \zeta X^{-1} e \right) = 0$$

complementary slackness conditions

The minimizer is positive, i.e., $x(\zeta) > 0$

so that it must satisfy $\nabla F(x) - A^T y - \zeta X^{-1} e = 0$

together with $Ax = b$

Optimality conditions for $\Phi(\zeta)$ are

$$\begin{cases} \nabla F(x) - A^T y - \zeta X^{-1} e = 0 \\ Ax = b \end{cases}$$

Define

$$s \equiv \nabla F(x) - yA^T$$

so that the first condition is

$$s - \zeta X^{-1} e = 0$$

$$\Rightarrow \begin{cases} X s = \zeta e \\ A x = b \end{cases}$$

Optimality Conditions

$$X s = \zeta e \quad \Rightarrow \quad x_j s_j = \zeta \quad \forall \; j=1,\ldots n$$

$$x_j s_j = 0$$

Complementary Slackness condition for the problem Φ^*

i.e.,

ζ is the violation in each complementary slackness condition of the problem Φ^*

$$\begin{cases} X^k s^k = \zeta^k e \\ A x^k = b \end{cases}$$

$$s^k = \nabla F(x^k) - A^T y^k$$

We want to use an iteration of the Newton-Raphson method to solve the nonlinear system

$$\begin{cases} X \;\; s \; = \hat{\zeta} \;\; e \\ A \; x \;\; = b \end{cases} \qquad \hat{\zeta} < \zeta^k$$

$$s = \nabla F(x) - A^T y$$

$$X s \approx X^k s^k + X^k \Delta s + S^k \Delta x$$

$$\Delta s \approx \nabla^2 F(x^k) \Delta x - A^T \Delta y$$

$$X s \approx X^k s^k + X^k \left(\nabla^2 F(x^k) \Delta x - A^T \Delta y\right) + S^k \Delta x$$

Setting $X s = \hat{\zeta} e$ gives

$$X^k s^k + X^k \left(\nabla^2 F(x^k) \Delta x - A^T \Delta y\right) + S^k \Delta x = \hat{\zeta} e$$

$$X^k s^k + X^k \left(\nabla^2 F(x^k) \Delta x - A^T \Delta y\right) + S^k \Delta x = \hat{\zeta} e$$

Since $x^k > 0$, X^k is nonsingular, and

$$s^k + \nabla^2 F(x^k)\, Dx - A^T \Delta y + (X^k)^{-1} S^k Dx = \hat{\zeta} (X^k)^{-1} e$$

Newton-Raphson step is found by solving the linear system:

$$\begin{cases} \left[\nabla^2 F(x^k) + (X^k)^{-1} S^k\right] \Delta x - A^T \Delta y = \hat{\zeta} (X^k)^{-1} e - s^k \\ A \Delta x = 0 \end{cases}$$

and

$$\begin{cases} x^{k+1} = x^k + \Delta x \\ y^{k+1} = y^k + \Delta y \\ s^{k+1} = \nabla F(x^{k+1}) - A^T y^{k+1} \\ z^{k+1} = \frac{1}{n} x^{k+1} s^{k+1} \end{cases}$$

If we reduce the barrier parameter by

$$\hat{\zeta} = \omega \; \zeta$$

for some factor ω, $0 < \omega < 1$

then we can perform another Newton-Raphson step to solve (approximately) the new nonlinear system of equations:

$$\begin{cases} X \; s = \hat{\zeta} \; e \\ A \; x = b \end{cases}$$

Geometric Programming Primal Problem

$$\begin{cases} \text{Minimize } g_0(t) \\ \text{subject to } g_k(t) \le 1, \; k=1,\cdots K \\ t_i > 0 \;, \; i=1,2,\cdots N \end{cases}$$

where $g_k(t) = \sum_{j\in[k]} c_j \prod_{i=1}^{N} t_i^{a_{ij}}$ *posynomial*

$$c_j > 0$$

$$\bigcup_k [k] = \{1,2,\cdots N\} \quad \& \quad [k'] \cap [k''] = \varnothing \quad \text{for } k' \neq k''$$

Application to Geometric Programming Dual Problem

The GP dual problem is linearly-constrained, and (if the negative of the log of the objective is minimized) has a convex objective function.

Geometric Programming Dual Problem

DGP: Maximize $v(\delta,\lambda) = \prod_{k=0}^{K} \left\{ \lambda_k^{\lambda_k} \prod_{i\in[k]} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \right\}$

subject to

$$\sum_{i\in[k]} \delta_i = \lambda_k,\ k=0,1,\cdots K$$

$$\sum_{i=1}^{N} a_{ij}\,\delta_i = 0,\ \ j=1,\cdots M$$

$$\lambda_0 = 1$$

$$\delta_i \ge 0,\ \lambda_k \ge 0\ \ \forall i,k$$

Note: $\bigcup_k [k] = \{1,2,\cdots N\}$ & $[k'] \cap [k''] = \varnothing$ for $k' \ne k''$

$$\text{Maximize } v(\delta,\lambda) = \prod_{k=0}^{K} \left\{ \lambda_k^{\lambda_k} \prod_{i\in[k]} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \right\}$$

is equivalent to

$$\text{Max } \ln v(\delta,\lambda) = \sum_{i=1}^{N} \{\delta_i \ln c_i - \delta_i \ln \delta_i\} + \sum_{k=0}^{K} \lambda_k \ln \lambda_k$$

or

$$\text{Min } -\ln v(\delta,\lambda) = \sum_{i=1}^{N} \{\delta_i \ln \delta_i - \delta_i \ln c_i\} - \sum_{k=0}^{K} \lambda_k \ln \lambda_k$$

$$\text{Min } -\ln v(\delta,\lambda) = \sum_{i=1}^{N} \{\delta_i \ln \delta_i - \delta_i \ln c_i\} - \sum_{k=0}^{K} \lambda_k \ln \lambda_k$$

The above objective is convex if we make the substitution $\sum_{i\in[k]} \delta_i = \lambda_k$

$$\text{Min } -\ln V(\delta) = \sum_{i=1}^{N} \{\delta_i \ln \delta_i - \delta_i \ln c_i\} - \sum_{k=0}^{K} \left[\sum_{i\in[k]} \delta_i\right] \ln \left[\sum_{i\in[k]} \delta_i\right]$$

DGP': Min $\sum_{i=1}^{N} \{\delta_i \ln \delta_i - \delta_i \ln c_i\} - \sum_{k=0}^{K} \left[\sum_{i\in[k]} \delta_i\right] \ln \left[\sum_{i\in[k]} \delta_i\right]$

subject to

$$\sum_{i\in[k]} \delta_i = 1 \qquad \textit{normality}$$

Geometric Programming Dual Problem

$$\sum_{i=1}^{N} a_{ij}\,\delta_i = 0,\ \ j=1,\cdots M \qquad \textit{orthogonality}$$

$$\delta_i \ge 0,\ \ \forall i$$

DGP' has several noteworthy properties:

- objective is convex
- constraints are linear
- if primal constraint k is slack
  $\Rightarrow \lambda_k = 0 \Rightarrow \sum_{i\in[k]} \delta_i = 0 \Rightarrow \delta_i = 0\ \forall i \in [k]$

- terms $\delta \ln \delta$ are difficult to compute for small positive
- while we may define $0 \ln 0 = \lim_{\delta\to 0} \delta \ln \delta = 0$

  $\delta \ln \delta$ is not differentiable at 0
- the objective is infinitely differentiable at positive

The algorithm which we have described was implemented in an experimental APL code PFAGP1.

$$x \longleftrightarrow \delta$$
$$y \longleftrightarrow \ln t$$
$$F(x) \longleftrightarrow -\ln V(\delta)$$

The primal GP variables (t) are obtained by exponentiating y

Obtaining the starting solution (x^0, s^0, y^0)

$$\begin{cases} X^0\, s^0 = \zeta^0\, e \\ A\, x^0 = b \qquad s^0 = \nabla F(x^0) - A^T y^0 \end{cases}$$

is accomplished by arbitrarily choosing x & y, & adding

an artificial variable x_{N+1} with large cost,

a bounding constraint $\sum_{i=1}^{N+1} x_i \le U$ with large U

setting $\zeta^0 = \frac{1}{N+1} \sum_{i=1}^{N+1} x_i^0 s_i^0$

Computational Experience

Beck & Ecker's Problem #10 will be used to illustrate the behavior of this algorithm

- # variables = 7
- # constraints = 4
- # terms = 20
- # "degrees of difficulty" = 12

There are 6 different variations, in which the exponent α in one term is varied.

As the exponent α is varied from $-1/4$ to $+1$, the first & last constraints change from being both slack to both active.

The GP dual problem is rather badly conditioned. For example, in the case $\alpha = +1$, the optimal values of three of the positive dual variables are less than 0.001, and most dual-based GP algorithms experience difficulty in determining the optimal primal variables.

B&E #10A

iteration	z	ω	z'/z
1	2.278E0	94.05	102.17
2	2.128E0	84.88	93.41
3	1.734E0	76.60	81.51
4	1.266E0	69.14	72.96
5	8.326E¯1	62.39	65.79
6	4.953E¯1	56.31	59.49
7	2.679E¯1	50.82	54.08
8	1.331E¯1	45.87	49.68
9	5.998E¯2	41.39	45.08
10	2.461E¯2	37.36	41.03
11	9.178E¯3	33.72	37.29
12	3.111E¯3	30.43	33.89
13	9.550E¯4	27.46	30.70
14	2.643E¯4	24.78	27.68
15	6.585E¯5	22.37	24.91
16	1.460E¯5	20.19	22.18
17	2.846E¯6	18.22	19.48
18	4.939E¯7	16.44	17.36
19	7.718E¯8	14.84	15.63
20	1.088E¯8	13.39	14.10
21	1.384E¯9	12.09	12.72

```
***Warning: Δx results in x+Δx<0 at iteration 1
   (x[2] = 0 at x + 0.53752306×Δx)
   Step used is 0.48377075×Δx
```


B&E #10A

iteration	z	ω	z'/z
⋮	⋮	⋮	⋮
22	1.589E¯10	10.91	11.48
23	1.647E¯11	9.84	10.36
24	3.208E¯12	14.41	19.48

```
***Deleting artificial variable, which is 5.4356152E¯14 at iteration 25
```

iteration	z	ω	z'/z
25	5.435E¯13	16.10	16.94
26	8.313E¯14	14.53	15.30
27	1.147E¯14	13.11	13.79
28	1.420E¯15	11.83	12.38

Converged in iteration # 27

B&E #10A

i	x	s	xs
1	5.5605774E¯1	2.6645353E¯15	1.4816354E¯15
2	4.4336474E¯1	2.6645353E¯15	1.1813610E¯15
3	2.3683684E¯4	6.0289551E¯12	1.4278787E¯15
4	3.4068164E¯4	4.1922021E¯12	1.4282063E¯15
5	6.2970015E¯16	2.2683319E0	1.4283689E¯15
6	6.1879025E¯15	2.3083249E¯1	1.4283689E¯15
7	2.6972464E¯15	5.2956560E¯1	1.4283689E¯15
8	6.1263254E¯1	1.9984014E¯15	1.2242858E¯15
9	1.3796628E0	1.1934898E¯15	1.6466135E¯15
10	8.7790268E¯2	1.5987212E¯14	1.4035216E¯15
11	1.0723557E0	1.1102230E¯15	1.1905540E¯15
12	4.3822934E¯1	3.1086245E¯15	1.3622904E¯15
13	7.9114982E¯1	1.9984014E¯15	1.5810349E¯15
14	2.2201890E¯1	7.3274720E¯15	1.6268373E¯15
15	3.3192332E¯16	4.3033099E0	1.4283689E¯15
16	2.3568874E¯15	6.0604037E¯1	1.4283689E¯15
17	5.1921119E¯16	2.7510365E0	1.4283689E¯15
18	1.2589491E¯15	1.1345724E0	1.4283689E¯15

```
Dual GP objective = 7.5009522          Objective function: 1809.7648
                  = log 1809.7648)      (gap = ¯6.0878619E¯8 i.e., ¯3.3638968E¯9%)
```


B&E #10A

Dual variables (of the dual GP problem) = y:

i	y[i]
1	¯7.5009522E0
2	1.0494776E0
3	¯4.9294800E¯1
4	7.6584571E¯1
5	1.5502979E0
6	¯5.1259050E¯4
7	2.9825659E¯1
8	¯3.4529297E0

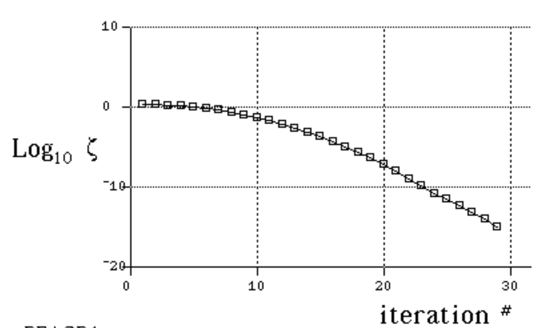
i	t[i]
1	2.8561586E0
2	6.1082303E¯1
3	2.1508126E0
4	4.7128737E0
5	9.9948754E¯1
6	1.3475075E0
7	3.1652767E¯2

z = 1.4196001E¯15

Constraints:

Posy	Value	Infeasibility	Lambda
2	0.6900658082	0.0000000E0	9.5148491E¯15
3	1.0000000000	0.0000000E0	2.0800857E0
4	1.0000000000	0.0000000E0	2.5237537E0
5	0.3868776663	0.0000000E0	4.4669711E¯15

B&E #10A



PFAGP1

B&E #10F

i	x	s	xs
1	7.2559669E¯2	1.9539925E¯14	1.4178105E¯15
2	9.1462207E¯1	4.4408921E¯16	4.0617379E¯16
3	1.2262304E¯3	1.1759482E¯12	1.4419835E¯15
4	1.1592026E¯2	1.2345680E¯13	1.4311144E¯15
5	1.5132990E¯3	9.5257136E¯13	1.4415253E¯15
6	7.1002941E¯2	2.0733415E¯14	1.4721334E¯15
7	1.1267145E¯2	1.2789769E¯13	1.4410418E¯15
8	2.7919468E¯1	5.5511151E¯15	1.5498418E¯15
9	9.0549382E¯1	1.3600232E¯15	1.2314926E¯15
10	1.3509246E¯1	1.0214052E¯14	1.3798414E¯15
11	1.1993180E0	8.8817842E¯16	1.0652084E¯15
12	6.1871705E¯1	2.2204460E¯15	1.3738278E¯15
13	4.8434933E¯1	2.8865799E¯15	1.3981130E¯15
14	1.5302901E¯1	9.3258734E¯15	1.4271292E¯15
15	2.3148566E¯4	6.2279071E¯12	1.4416712E¯15
16	3.3710643E¯1	4.4408921E¯15	1.4970533E¯15
17	2.6793648E¯4	5.3823612E¯12	1.4421309E¯15
18	1.0659804E¯2	1.3544721E¯13	1.4438406E¯15

Dual GP objective = 6.0212145
= log 412.07877)

z = 1.3501074E¯15
Objective function: 412.07877
(gap = ¯1.1523923E¯8 i.e., ¯2.7965341E¯9%)

PFAGP1

B&E #10F

Dual variables (of the dual GP problem) = y:

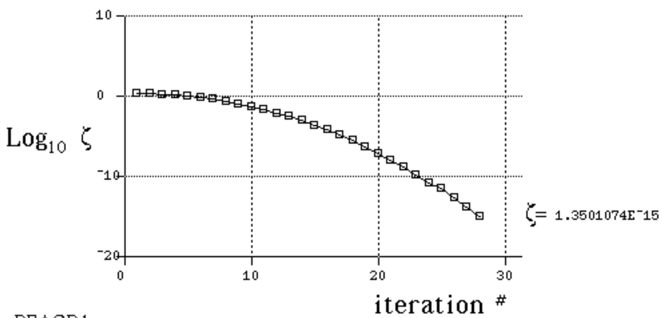
i	y[i]
1	¯6.0212145E0
2	1.5098922E0
3	¯1.8874188E¯1
4	1.0374129E0
5	1.0785433E0
6	¯4.4331639E¯1
7	¯2.7789153E¯1
8	¯3.5941118E0

i	t[i]
1	4.5262428E0
2	8.2800020E¯1
3	2.8219069E0
4	2.9403930E0
5	6.4190409E¯1
6	7.5737897E¯1
7	2.7485085E¯2

Constraints:

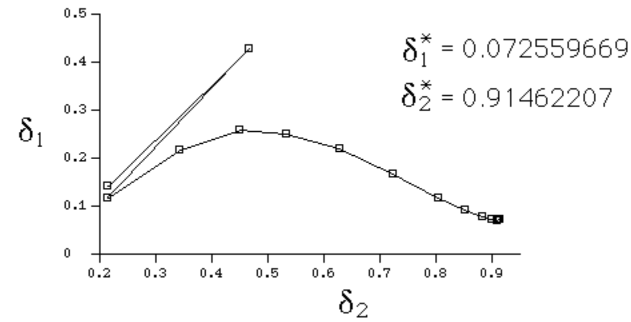
Posy	Value	Infeasibility	Lambda
2	1.0000000000	0.0000000E0	8.3783385E¯2
3	1.0000000000	0.0000000E0	1.3197810E0
4	1.0000000000	0.0000000E0	2.4554134E0
5	1.0000000000	0.0000000E0	3.4826565E¯1

B&E #10F



PFAGP1

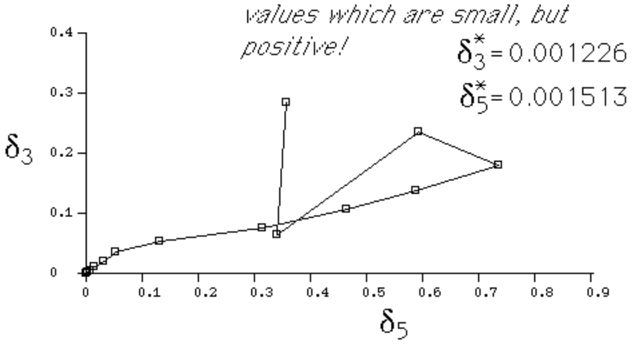
B&E #10F



PFAGP1

B&E #10F

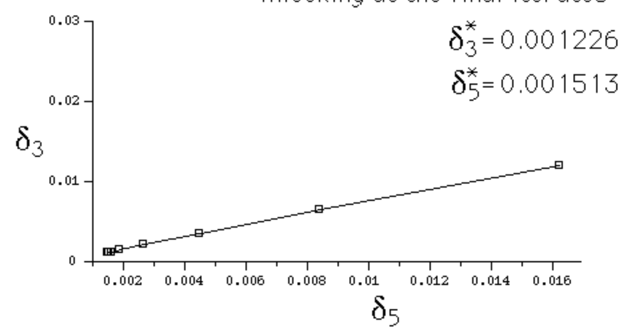
These two variables have optimal values which are small, but positive!



PFAGP1

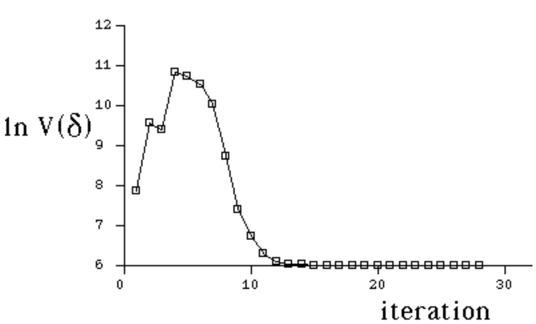
B&E #10F

...looking at the final iterates



PFAGP1

B&E #10F



PFAGP1

Dual-based GP algorithms generally have difficulty with problems such as the following:

Minimize $g_0(t) = t_1 t_2 + t_1^{-1} t_2^{-1}$
subject to
$$g_1(t) = \frac{1}{4} t_1^{0.5} + t_2 \le 1$$
$$t_1 > 0,\ t_2 > 0$$

The dual problem has a unique solution

$$\delta_1^* = \delta_2^* = 1/2,\ \lambda_1 = \delta_3^* = \delta_4^* = 0$$

The primal solution, which is not unique, cannot be determined by solving the equations

$$c_j \prod_i x_i^{a_{ij}} = \frac{\delta_j}{\lambda_k} \text{ where } j \in [k]$$

i	x	s	xs
1	5.0000000E¯1	1.8651747E¯14	9.3258734E¯15
2	5.0000000E¯1	1.8429702E¯14	9.2148511E¯15
3	4.2349134E¯14	2.3035641E¯1	9.7553946E¯15
4	2.1108084E¯14	4.0233010E¯1	8.4924176E¯15
5	9.0000000E0	1.0144972E¯15	9.1304747E¯15
6	1.0227926E¯15	8.9478720E0	9.1518177E¯15

(Artificial variable: 1.0227926E¯15)
(Slack variable: 9)
z = 9.1784715E¯15

Dual GP objective = 0.69314718
= log 2)

Objective function: 2
(gap = 3.6193271E¯14 i.e.,
1.8096635E¯12%)

Dual variables (of the dual GP problem) = y:

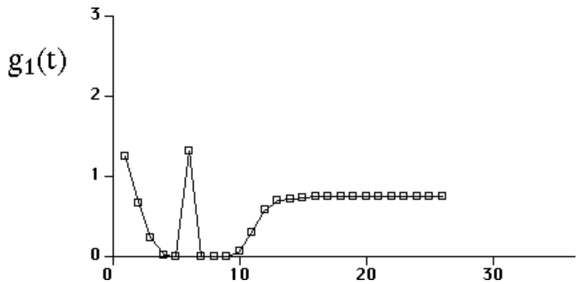
i	y[i]
1	¯6.9314718E¯1
2	1.5030399E0
3	¯1.5030399E0
4	¯1.0144972E¯15

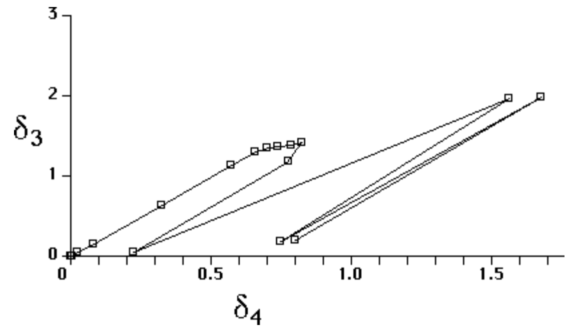
Primal GP solution (t):

i	t[i]
1	4.4953339E0
2	2.2245289E¯1

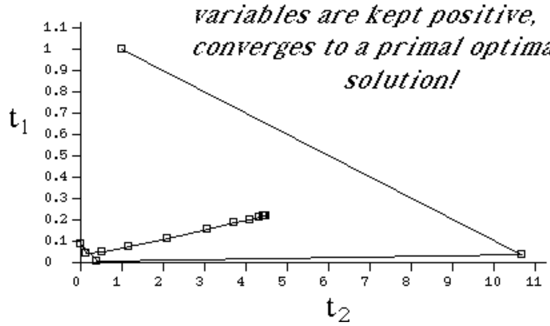
Constraints:

Posy	Value	Infeasibility	Lambda
2	0.7525079477	0.0000000E0	6.3457218E¯14





The algorithm, because dual variables are kept positive, converges to a primal optimal solution!



$$\text{Max } \ln v(\delta,\lambda) = \sum_{i=1}^{N} \{\delta_i \ln c_i - \delta_i \ln \delta_i\} + \sum_{k=0}^{K} \lambda_k \ln \lambda_k$$

subject to

$$\sum_{i\in[k]} \delta_i = \lambda_k,\ k=0,1,\cdots K$$

$$\sum_{i=1}^{N} a_{ij}\,\delta_i = 0,\ j=1,\cdots M$$

orthogonality

$$\lambda_0 = 1$$

$$\delta_i \geq 0,\ \lambda_k \geq 0$$

Geometric Programming Dual Problem

Make a change of variable:

$$\delta_j = \rho_j \lambda_k \text{ for } j \in [k]$$

so that $\rho_j = \dfrac{\delta_j}{\lambda_k}$ if $j \in [k]$ & $\lambda_k > 0$

Define functions

$$G_k(\rho) \equiv \sum_{j\in[k]} \{\rho_j \ln c_j - \rho_j \ln \rho_j\}$$

entropy function

$$A_{ki}(\rho) \equiv \sum_{j\in[k]} a_{ij}\,\rho_j$$

Maximize $\sum_{k=0}^{P} G_k(\rho)\,\lambda_k$

subject to

$$\sum_{k=0}^{P} A_{ki}(\rho)\,\lambda_k = 0,\ i=1,\ldots N$$

$$\lambda_0 = 1$$

Geometric Programming Dual Problem

$$\sum_{j\in[k]} \rho_j = 1,\ k=0,1,\ldots K$$

$$\lambda_k \ge 0,\ \rho_j \ge 0,\ \forall k,j$$

For fixed values of λ, this is an entropy problem....

For fixed values of ρ, this is an LP problem.

The path-following algorithm has polynomial-time complexity for both entropy & LP problems.

If t is optimal in the primal, and (ρ, λ) is optimal in the dual,

then

$$\rho_j > 0 \text{ and } g_k(t) > 0$$

and

$$\rho_j = \frac{c_j \prod_{i=1}^{N} t_i^{a_{ij}}}{g_k(t)}$$

whether the constraint k is tight or slack!