Generalized Linear Programming (GLP) Formulation of the Geometric Programming Dual Problem

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Standard Dual of Posynomial GP

$$\text{Max} \quad \ln v(\delta,\lambda) = \sum_{i=1}^{N} \left\{ \delta_i \ln c_i - \delta_i \ln \delta_i \right\} + \sum_{k=0}^{K} \lambda_k \ln \lambda_k$$

subject to

$$\sum_{i\in[k]} \delta_i = \lambda_k \,,\ k=0,1,\cdots K$$

$$\sum_{i=1}^{N} a_{ij}\, \delta_i = 0,\ \ j=1,\cdots M$$

orthogonality

$$\lambda_0 = 1$$

$$\delta_i \ge 0,\ \lambda_k \ge 0$$

Undesirable Properties of GP Dual:

- objective function is nondifferentiable if $\delta_i = 0$ & $\lambda_k = 0$ for any i & k
- if λ_k=0, then $\delta_i = 0\ \forall\, i \in [k]$
- it is possible that the dual solution does not provide sufficient information to compute the optimal primal solution.
- for small δ_i, computation of the terms $\delta_i \ln \delta_i$ introduce substantial numerical errors.

Make a change of variable:

$$\boxed{\delta_j = \rho_j \lambda_k \ \ \text{for } j \in [k]}$$

so that $\rho_j = \dfrac{\delta_j}{\lambda_k}$ if $j \in [k]$ & $\lambda_k > 0$

Note that

$$\sum_{i\in[k]} \delta_i = \lambda_k \implies \sum_{j\in[k]} \rho_j = 1 \quad \forall k$$

Define functions

$$G_k(\rho) \equiv \sum_{j\in[k]} \left\{ \rho_j \ln c_j - \rho_j \ln \rho_j \right\}$$

entropy function

$$A_{ki}(\rho) \equiv \sum_{j\in[k]} a_{ij}\, \rho_j$$

$$\text{Maximize} \sum_{k=0}^{P} G_k(\rho)\, \lambda_k$$

subject to

$$\sum_{k=0}^{P} A_{ki}(\rho)\, \lambda_k = 0,\ i=1,\ldots N$$

$$\lambda_0 = 1$$

Geometric Programming Dual Problem

$$\sum_{j\in[k]} \rho_j = 1,\ k=0,1,\ldots K$$

$$\lambda_k \ge 0,\ \rho_j \ge 0,\ \ \forall k,j$$

For fixed values of λ, this is an entropy problem....

For fixed values of ρ, this is an LP problem.

That is,

$$\underset{\gamma,\lambda}{\text{Maximize}} \sum_{k=0}^{P} \gamma_k \lambda_k$$

subject to

$$\sum_{k=0}^{P} \alpha_{kj} \lambda_k = 0,\ j=1,\cdots m$$

$$\lambda_0 \equiv 1$$

$$\lambda_k \ge 0,\ k=1,\cdots p$$

$$(\gamma_k,\alpha_k) \in S_k\ ,\ k=0,\cdots p$$

Linear Program in λ

where

$$S_k = \left\{ (\gamma,\alpha) \mid \exists\, \rho \ge 0,\ \sum_{i\in[k]} \rho_i = 1 \text{ such that } \gamma = G_k(\rho),\ \alpha_j = A_{kj}(\rho) \right\}$$

This class of problems, which are linear programming problems in which the columns are also to be selected, was called

Generalized Linear Programs

by George Dantzig, and was used in solving chemical equilibrium problems.

If t is optimal in the primal, and (ρ, λ) is optimal in the dual,

then $\rho_j > 0$ and $g_k(t) > 0$

and

$$\boxed{\rho_j = \frac{c_j \prod_{i=1}^{N} t_i^{a_{ij}}}{g_k(t)}}$$

whether the constraint k is tight or slack!

The dual solution thereby always provides sufficient information to compute the primal solution!

Column-Generating Algorithm

S_k is a convex set, and is therefore the convex hull of its set of extreme points,

$$\mathrm{ext}(S_k) = \{(\hat{\gamma}_k^n, \hat{\alpha}_k^n)\}_{n \in N_k}$$

index set of extreme points

For simplicity, we assume that the set of extreme points is countable!

⇦ ⇨

$$\boxed{(\gamma_k, \alpha_k) \in S_k} \Leftrightarrow \boxed{\begin{array}{ll} & \exists\, \mu_{kn} \geq 0,\ n \in N_k \\ \text{such that} & \sum_{n \in N_k} \mu_{kn} = 1 \\ & \gamma_k = \sum_{n \in N_k} \mu_{kn} \hat{\gamma}_k^n \\ & \alpha_k = \sum_{n \in N_k} \mu_{kn} \hat{\alpha}_k^n \end{array}}$$

Any element of a convex set can be represented by a convex combination of the extreme points of the set!

⇦ ⇨

The GLP dual may then be written

$$\text{Maximize } \sum_{k=0}^{p} \left[\sum_{n \in N_k} \mu_{kn} \hat{\gamma}_k^n \right] \lambda_k$$

$$\sum_{k=0}^{p} \left[\sum_{n \in N_k} \mu_{kn} \hat{\alpha}_k^n \right] \lambda_k = 0,\ j=1, \cdots m$$

$$\left[\sum_{n \in N_0} \mu_{0n} \right] \lambda_0 = 1$$

⇦ ⇨

or, by defining $u_{kn} \equiv \mu_{kn} \lambda_k$

$$\text{Maximize } \sum_{k=0}^{p} \sum_{n \in N_k} \hat{\gamma}_k^n u_{kn}$$

$$\text{subject to } \sum_{k=0}^{p} \sum_{n \in N_k} \hat{\alpha}_k^n u_{kn} = 0,\ j=1, \cdots m$$

$$\sum_{n \in N_0} u_{0n} = 1$$

which is an "ordinary" LP with infinitely many variables *(semi-infinite LP)*

⇦ ⇨

Step 0 For each k, select one or more extreme points of

Step 1 Generate an LP with columns and variables corresponding to the sets of extreme points

Step 2 Compute the simplex multipliers of the orthogonality constraints and normality constraint.

⇦ ⇨ Column-Generating Algorithm for SILP

Step 3 For each k=0,1,...p, choose (γ_k, α_k) so as to maximize the relative profit:

$$\boxed{\underset{(\gamma,\alpha) \in S_0}{\text{maximize}} \ \gamma - \sum_{j=1}^{m} w_j \alpha_j - w_{m+1}} \quad \text{if } k=0$$

$$\boxed{\underset{(\gamma,\alpha) \in S_k}{\text{maximize}} \ \gamma - \sum_{j=1}^{m} w_j \alpha_j} \quad \text{if } k>0$$

⇦ ⇨ Column-Generating Algorithm for SILP

Step 4

For each (γ_k, α_k) whose relative profit exceeds some tolerance, add the corresponding column to the LP tableau.

If no columns can be added, stop; else, return to **Step 1**

⇦ ⇨ **Column-Generating Algorithm for SILP**

Maximizing the Relative Profit

For each k, the column having maximum relative profit is (γ_k, α_k) where

$$\gamma_k = G_k(\rho), \quad \alpha_k = A_{k\,j}(\rho)$$

and
$$\rho_n = \frac{c_n\, e^{\sum_j a_{nj} w_j}}{\sum_{i \in [k]} c_i\, e^{\sum_j a_{ij} w_j}}$$

⇦ ⇨

Maximizing the Relative Profit

That is, if we compute the primal (approximate) solution $x_j = e^{w_j}, j = 1, \dots m$

(i.e., exponentiate the simplex multipliers of the orthogonality conditions)

then
$$\boxed{\rho_n = \frac{c_n \prod_j x_j^{a_{nj}}}{\sum_{i \in [k]} c_i \prod_j x_j^{a_{ij}}}} = \frac{\textit{term } n}{\textit{posynomial } k} > 0$$

$$\forall\ n \in [k]$$

⇦ ⇨

For the ρ thus obtained, i.e.,

$$\boxed{\rho_n = \frac{c_n \prod_j x_j^{a_{nj}}}{\sum_{i \in [k]} c_i \prod_j x_j^{a_{ij}}}} \quad \forall\ n \in [k]$$

the relative profit function is nonpositive if & only if

$$g_0(x) = g_0(x^*)$$

$$g_k(x) \le 1$$

⇦ ⇨

Multi-Item EOQ **EXAMPLE**

Demand for three items is known, with the rate of demand constant over time, D_i/year. Holding cost of item #i is H_i /unit/year, and each replenishment incurs a cost of A_i.

Let Q_i = order quantity of item #i.

D_i/Q_i = average # replenishments per year

$Q_i/2$ = average inventory level

⇦ ⇨

Annual cost of item #i is
$$\underbrace{\frac{A_i D_i}{Q_i}}_{\textit{ordering cost}} + \underbrace{\frac{1}{2} H_i\, Q_i}_{\textit{holding cost}}$$

The classic EOQ ("economic order quantity") formula of Wilson specifies the order quantity which minimizes this annual cost:

$$\boxed{Q_i^* = \sqrt{\frac{2 A_i D_i}{H_i}}}$$

⇦ ⇨

Often there are additional constraints on the ordering policy, e.g.,

- a limit on the number of replenishments/year
$$\sum_i \frac{D_i}{Q_i} \le \overline{N}$$
- a limit on the maximum volume (if all orders were to arrive simultaneously)
$$\sum_i v_i Q_i \le \overline{V}$$
- a limit on the average investment in inventory
$$\sum_i \tfrac{1}{2} C_i Q_i \le \overline{B}$$

⇦ ⇨

EXAMPLE

i	A_i	D_i	H_i
1	50	1000	4
2	100	2000	5
3	80	2000	3

$\widehat{Q} = 100$

$$\text{Minimize } \sum_{i=1}^{3} \left[\frac{A_i D_i}{Q_i} + \frac{1}{2} H_i\, Q_i \right]$$

$$\text{subject to } \sum_{i=1}^{3} Q_i \le \widehat{Q}$$

$$Q_i > 0,\ i = 1,2,3$$

Multi-Item EOQ

⇦ ⇨

```
Number of variables  : 3
Number of posynomials: 2
Total number of terms: 9
Degrees of difficulty: 5
Terms per posynomial: 6 3

Coefficients and exponent matrix:
                 ┌───┬───┬┬───────────┬──────────┐
                 │ t │ p ││    Ct     │ exponent │
                 ├───┼───┼┼───────────┼──────────┤
                 │ 1 │ 1 ││  50000    │ ¯1  0  0 │
                 │ 2 │ 1 ││      2    │  1  0  0 │
                 │ 3 │ 1 ││200000     │  0 ¯1  0 │
                 │ 4 │ 1 ││      2.5  │  0  1  0 │
                 │ 5 │ 1 ││160000     │  0 ¯1  0 │
                 │ 6 │ 1 ││      1.5  │  0  0  1 │
                 │ 7 │ 2 ││      0.01 │  1  0  0 │
                 │ 8 │ 2 ││      0.01 │  0  1  0 │
                 │ 9 │ 2 ││      0.01 │  0  0  1 │
                 └───┴───┴┴───────────┴──────────┘
                                 t = term number
                                 p = posynomial
                                 Ct = coefficient
```

```
        ┌────────────────┐
        │ Multi-Item EOQ │
        └────────────────┘

   ┌────────────┬────────────┐
   │ Posynomial │ Tolerance  │
   ├────────────┼────────────┤
   │          1 │ 5.00E¯4    │
   │          2 │ 5.00E¯4    │
   └────────────┴────────────┘
```

i.e., add a column for each posynomial constraint whose infeasibility exceeds 0.0005 and for the objective function if the duality gap exceeds 0.05%.

```
  Frequencies for
     Discarding unused grid pts: 10
     Reporting solutions: 1
  Types of grid points to be generated: 0 1
```

```
                          Iteration 1
                          -----------

                          LP Solution
                          -----------
Col  Posy  Type    Value
   1    0    0  8.27741236
  14    2    0  1.00000000
   6    1    0  0.42578125
   2    1    0  0.20572917
   4    1    0  0.36848958

Determinant of the basis matrix = ¯1
```

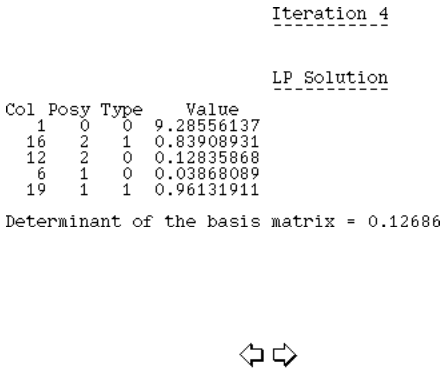
```
---------------Exponentiating LP Dual solution--------

X = 12.71 50.839 40.671
Weights (ρ):
0.32741 0.0021155 0.32741 0.010578 0.32741 0.0050773
0.12195 0.4878 0.39024

Objective functions:
  Primal: 12016   Dual: 3934
  Duality Gap: 8081.5 = 205.43 percent

Constraints:

                   Value         1.0422
                   Infeasibility 0.042196
                   Lambda        1


Type-1 grid points (#15 16) added for posynomials 1 2
CPU time: 34.6 sec.
```

```
                          Iteration 2
                          -----------

                          LP Solution
                          -----------
Col  Posy  Type    Value
   1    0    0  9.27177390
   4    1    0  0.00547033
  12    2    0  0.96170767
  15    1    1  0.99452967
   8    2    0  0.00294556

Determinant of the basis matrix = 0.33517
```

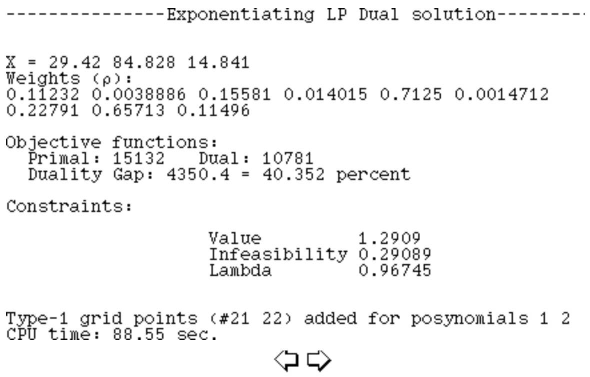
```
---------------Exponentiating LP Dual solution--------

X = 100 18.808 19.692
Weights (ρ):
0.025595 0.010238 0.54433 0.002407 0.41592 0.001512
0.72202 0.1358 0.14218

Objective functions:
  Primal: 19535   Dual: 10634
  Duality Gap: 8901.7 = 83.713 percent

Constraints:

                   Value         1.385
                   Infeasibility 0.385
                   Lambda        0.96465


Type-1 grid points (#17 18) added for posynomials 1 2
CPU time: 56.6 sec.
```

```
                          Iteration 3
                          -----------

                          LP Solution
                          -----------
Col  Posy  Type    Value
   1    0    0  9.27910603
  14    2    0  0.03474715
  12    2    0  0.92990237
  15    1    1  0.97361308
  17    1    1  0.02638692

Determinant of the basis matrix = 0.017434
```

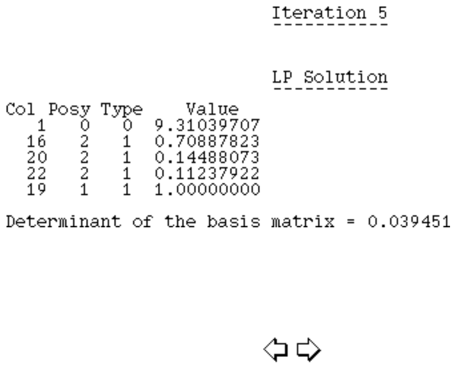
```
---------------Exponentiating LP Dual solution----------

X = 27.139 35.341 38.616
Weights (ρ):
0.15554 0.0045821 0.47775 0.0074588 0.34979 0.00489
0.26845 0.34958 0.38197

Objective functions:
  Primal: 11845   Dual: 10712
  Duality Gap: 1133.6 = 10.583 percent

Constraints:

                   Value         1.011
                   Infeasibility 0.010958
                   Lambda        0.96465


Type-1 grid points (#19 20) added for posynomials 1 2
```



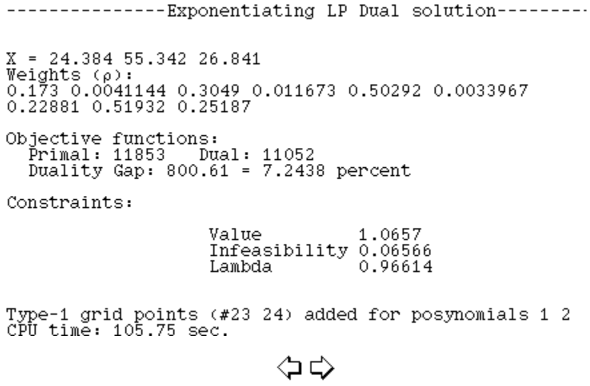
```
                    Iteration 4

                    LP Solution

Col Posy Type    Value
  1   0    0  9.28556137
 16   2    1  0.83908931
 12   2    0  0.12835868
  6   1    0  0.03868089
 19   1    1  0.96131911

Determinant of the basis matrix = 0.12686
```



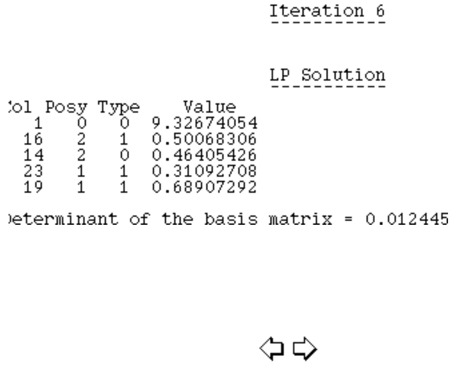
```
---------------Exponentiating LP Dual solution---------

X = 29.42 84.828 14.841
Weights (ρ):
0.11232 0.0038886 0.15581 0.014015 0.7125 0.0014712
0.22791 0.65713 0.11496

Objective functions:
  Primal: 15132   Dual: 10781
  Duality Gap: 4350.4 = 40.352 percent

Constraints:

                   Value         1.2909
                   Infeasibility 0.29089
                   Lambda        0.96745


Type-1 grid points (#21 22) added for posynomials 1 2
CPU time: 88.55 sec.
```



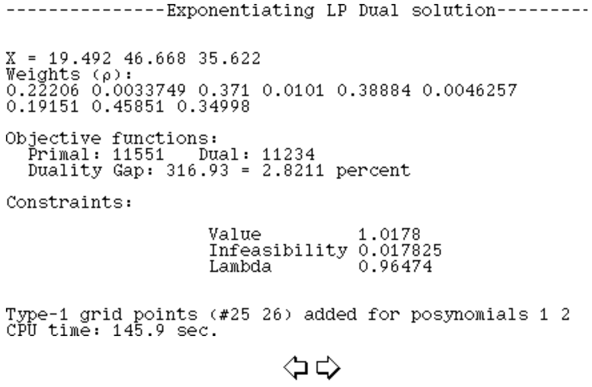
```
                      Iteration 5

                      LP Solution

Col Posy Type    Value
  1   0    0  9.31039707
 16   2    1  0.70887823
 20   2    1  0.14488073
 22   2    1  0.11237922
 19   1    1  1.00000000

Determinant of the basis matrix = 0.039451
```



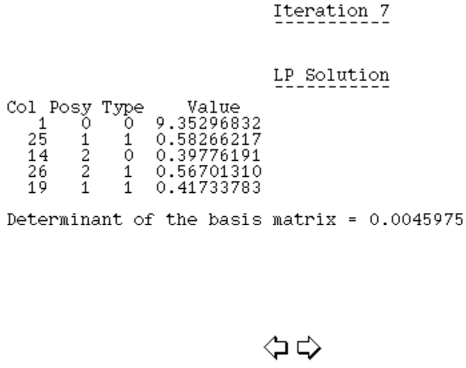
```
---------------Exponentiating LP Dual solution---------

X = 24.384 55.342 26.841
Weights (ρ):
0.173 0.0041144 0.3049 0.011673 0.50292 0.0033967
0.22881 0.51932 0.25187

Objective functions:
  Primal: 11853   Dual: 11052
  Duality Gap: 800.61 = 7.2438 percent

Constraints:

                   Value         1.0657
                   Infeasibility 0.06566
                   Lambda        0.96614


Type-1 grid points (#23 24) added for posynomials 1 2
CPU time: 105.75 sec.
```



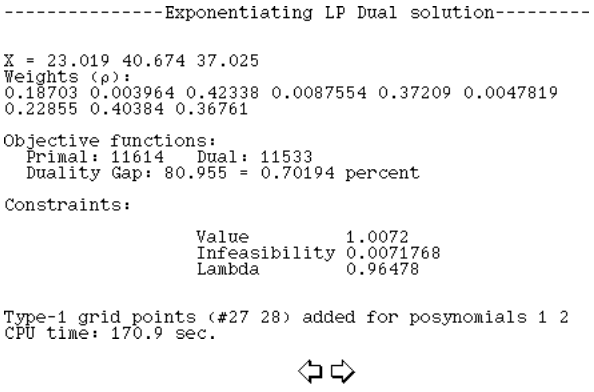
```
                      Iteration 6

                      LP Solution

ol Posy Type    Value
  1   0    0  9.32674054
 16   2    1  0.50068306
 14   2    0  0.46405426
 23   1    1  0.31092708
 19   1    1  0.68907292

eterminant of the basis matrix = 0.012445
```



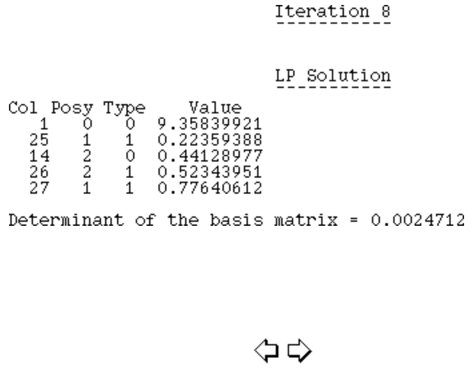
```
---------------Exponentiating LP Dual solution---------

X = 19.492 46.668 35.622
Weights (ρ):
0.22206 0.0033749 0.371 0.0101 0.38884 0.0046257
0.19151 0.45851 0.34998

Objective functions:
  Primal: 11551   Dual: 11234
  Duality Gap: 316.93 = 2.8211 percent

Constraints:

                   Value         1.0178
                   Infeasibility 0.017825
                   Lambda        0.96474


Type-1 grid points (#25 26) added for posynomials 1 2
CPU time: 145.9 sec.
```



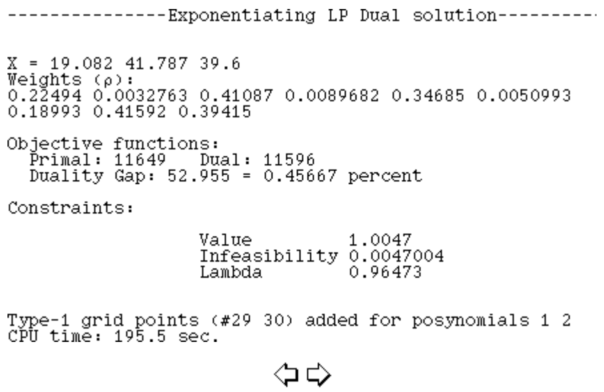
```
                      Iteration 7

                      LP Solution

Col Posy Type    Value
  1   0    0  9.35296832
 25   1    1  0.58266217
 14   2    0  0.39776191
 26   2    1  0.56701310
 19   1    1  0.41733783

Determinant of the basis matrix = 0.0045975
```



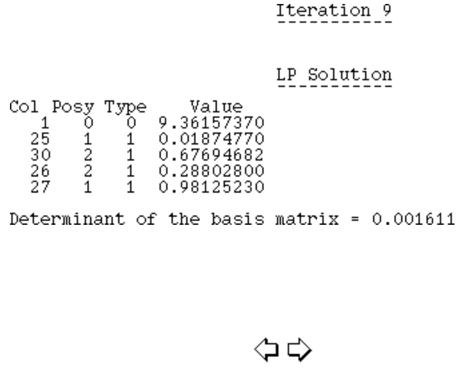
```
---------------Exponentiating LP Dual solution---------

X = 23.019 40.674 37.025
Weights (ρ):
0.18703 0.003964 0.42338 0.0087554 0.37209 0.0047819
0.22855 0.40384 0.36761

Objective functions:
  Primal: 11614   Dual: 11533
  Duality Gap: 80.955 = 0.70194 percent

Constraints:

                   Value         1.0072
                   Infeasibility 0.0071768
                   Lambda        0.96478


Type-1 grid points (#27 28) added for posynomials 1 2
CPU time: 170.9 sec.
```



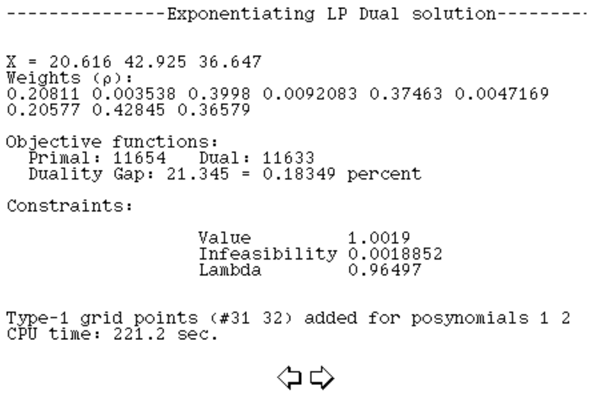
```
                    Iteration 8
                    -----------

                    LP Solution
                    -----------
Col Posy Type    Value
  1   0    0  9.35839921
 25   1    1  0.22359388
 14   2    0  0.44128977
 26   2    1  0.52343951
 27   1    1  0.77640612

Determinant of the basis matrix = 0.0024712
```



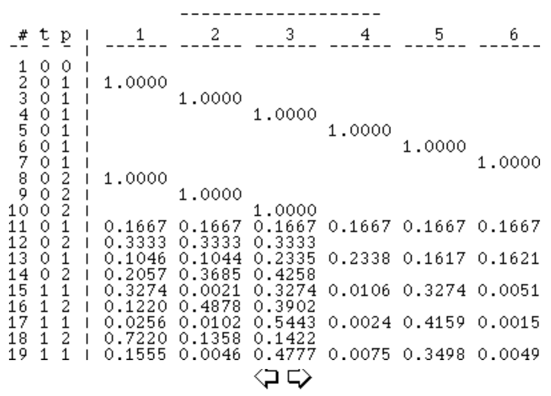
```
---------------Exponentiating LP Dual solution---------

X = 19.082 41.787 39.6
Weights (ρ):
0.22494 0.0032763 0.41087 0.0089682 0.34685 0.0050993
0.18993 0.41592 0.39415

Objective functions:
  Primal: 11649   Dual: 11596
  Duality Gap: 52.955 = 0.45667 percent

Constraints:

                  Value         1.0047
                  Infeasibility 0.0047004
                  Lambda        0.96473


Type-1 grid points (#29 30) added for posynomials 1 2
CPU time: 195.5 sec.
```



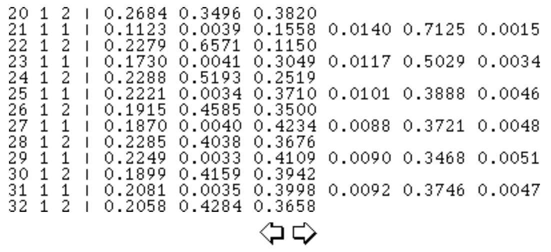
```
                    Iteration 9
                    -----------

                    LP Solution
                    -----------
Col Posy Type    Value
  1   0    0  9.36157370
 25   1    1  0.01874770
 30   2    1  0.67694682
 26   2    1  0.28802800
 27   1    1  0.98125230

Determinant of the basis matrix = 0.001611
```



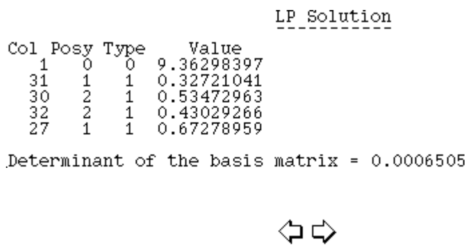
```
---------------Exponentiating LP Dual solution--------

X = 20.616 42.925 36.647
Weights (ρ):
0.20811 0.003538 0.3998 0.0092083 0.37463 0.0047169
0.20577 0.42845 0.36579

Objective functions:
  Primal: 11654   Dual: 11633
  Duality Gap: 21.345 = 0.18349 percent

Constraints:

                  Value         1.0019
                  Infeasibility 0.0018852
                  Lambda        0.96497


Type-1 grid points (#31 32) added for posynomials 1 2
CPU time: 221.2 sec.
```



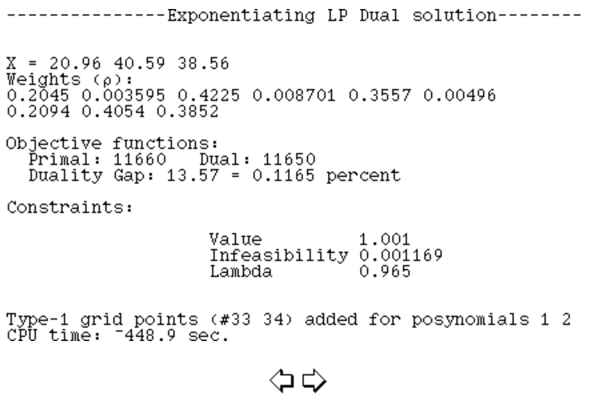
```
                   -------------------
 # t p |    1      2      3      4      5      6
-- - - | ------ ------ ------ ------ ------ ------
 1 0 0 |
 2 0 1 | 1.0000
 3 0 1 |        1.0000
 4 0 1 |               1.0000
 5 0 1 |                      1.0000
 6 0 1 |                             1.0000
 7 0 1 |                                    1.0000
 8 0 2 | 1.0000
 9 0 2 |        1.0000
10 0 2 |               1.0000
11 0 1 | 0.1667 0.1667 0.1667 0.1667 0.1667 0.1667
12 0 2 | 0.3333 0.3333 0.3333
13 0 1 | 0.1046 0.1044 0.2335 0.2338 0.1617 0.1621
14 0 2 | 0.2057 0.3685 0.4258
15 1 1 | 0.3274 0.0021 0.3274 0.0106 0.3274 0.0051
16 1 2 | 0.1220 0.4878 0.3902
17 1 1 | 0.0256 0.0102 0.5443 0.0024 0.4159 0.0015
18 1 2 | 0.7220 0.1358 0.1422
19 1 1 | 0.1555 0.0046 0.4777 0.0075 0.3498 0.0049
```



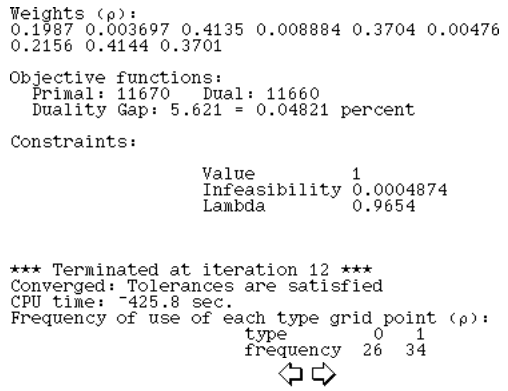
```
20 1 2 | 0.2684 0.3496 0.3820
21 1 1 | 0.1123 0.0039 0.1558 0.0140 0.7125 0.0015
22 1 2 | 0.2279 0.6571 0.1150
23 1 1 | 0.1730 0.0041 0.3049 0.0117 0.5029 0.0034
24 1 2 | 0.2288 0.5193 0.2519
25 1 1 | 0.2221 0.0034 0.3710 0.0101 0.3888 0.0046
26 1 2 | 0.1915 0.4585 0.3500
27 1 1 | 0.1870 0.0040 0.4234 0.0088 0.3721 0.0048
28 1 2 | 0.2285 0.4038 0.3676
29 1 1 | 0.2249 0.0033 0.4109 0.0090 0.3468 0.0051
30 1 2 | 0.1899 0.4159 0.3942
31 1 1 | 0.2081 0.0035 0.3998 0.0092 0.3746 0.0047
32 1 2 | 0.2058 0.4284 0.3658
```



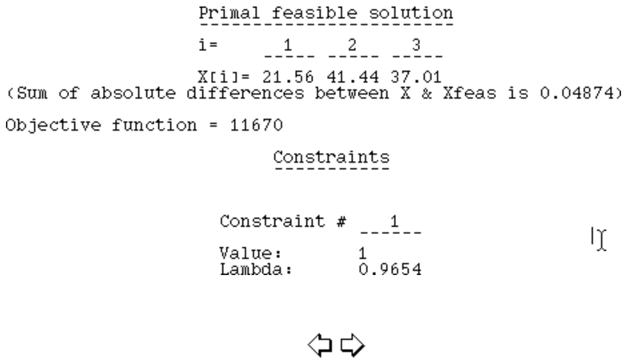
```
                    LP Solution
                    -----------
Col Posy Type    Value
  1   0    0  9.36298397
 31   1    1  0.32721041
 30   2    1  0.53472963
 32   2    1  0.43029266
 27   1    1  0.67278959

Determinant of the basis matrix = 0.0006505
```



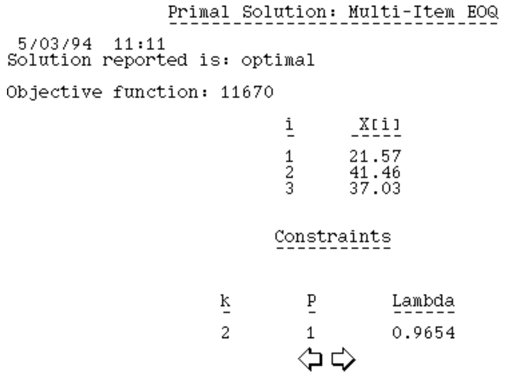
```
---------------Exponentiating LP Dual solution--------

X = 20.96 40.59 38.56
Weights (ρ):
0.2045 0.003595 0.4225 0.008701 0.3557 0.00496
0.2094 0.4054 0.3852

Objective functions:
  Primal: 11660   Dual: 11650
  Duality Gap: 13.57 = 0.1165 percent

Constraints:

                  Value         1.001
                  Infeasibility 0.001169
                  Lambda        0.965


Type-1 grid points (#33 34) added for posynomials 1 2
CPU time: ¯448.9 sec.
```



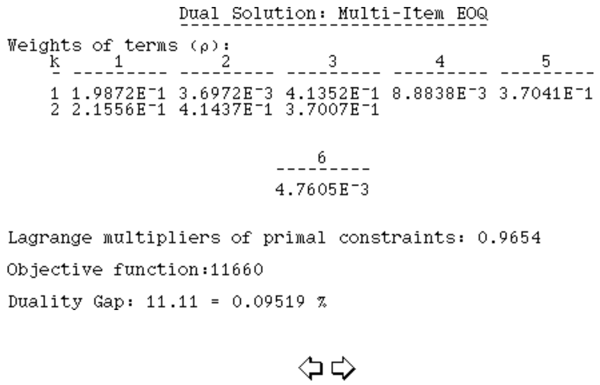
```
Weights (ρ):
0.1987 0.003697 0.4135 0.008884 0.3704 0.00476
0.2156 0.4144 0.3701

Objective functions:
  Primal: 11670   Dual: 11660
  Duality Gap: 5.621 = 0.04821 percent

Constraints:

                  Value          1
                  Infeasibility 0.0004874
                  Lambda         0.9654


*** Terminated at iteration 12 ***
Converged: Tolerances are satisfied
CPU time: ¯425.8 sec.
Frequency of use of each type grid point (ρ):
                      type        0   1
                      frequency  26  34
```



```
                  Primal feasible solution
                  ------------------------
                  i=      1     2     3
                        ----- ----- -----
                  X[i]= 21.56 41.44 37.01
(Sum of absolute differences between X & Xfeas is 0.04874)

Objective function = 11670

                        Constraints
                        -----------

                  Constraint #  __1__

                  Value:       1
                  Lambda:      0.9654
```

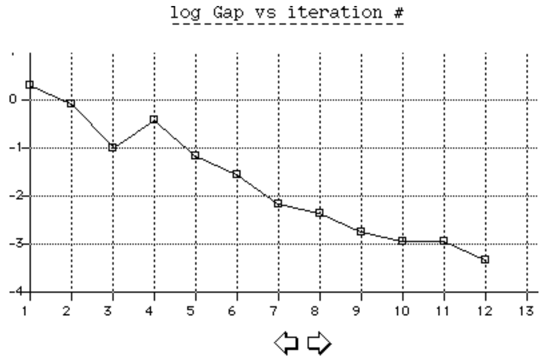


```
           Primal Solution: Multi-Item EOQ
           -------------------------------
 5/03/94  11:11
Solution reported is: optimal

Objective function: 11670

                              i      X[i]
                              -      ----
                              1     21.57
                              2     41.46
                              3     37.03

                             Constraints
                             -----------

                   k       P        Lambda
                   -       -        ------
                   2       1        0.9654
```

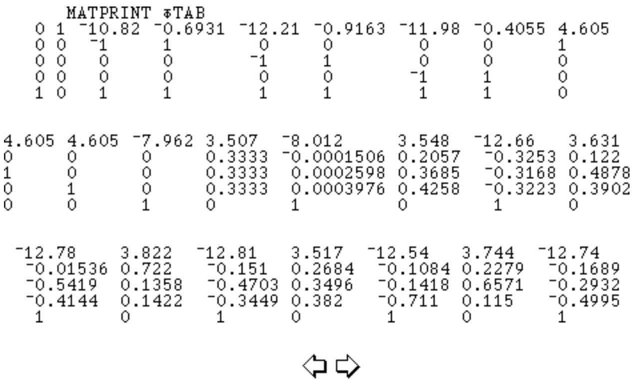


```
            Dual Solution: Multi-Item EOQ
            -----------------------------
Weights of terms (ρ):
   k     1         2         3         4         5
   - --------- --------- --------- --------- ---------
   1 1.9872E¯1 3.6972E¯3 4.1352E¯1 8.8838E¯3 3.7041E¯1
   2 2.1556E¯1 4.1437E¯1 3.7007E¯1

                         6
                     ---------
                     4.7605E¯3

Lagrange multipliers of primal constraints: 0.9654

Objective function:11660

Duality Gap: 11.11 = 0.09519 %
```

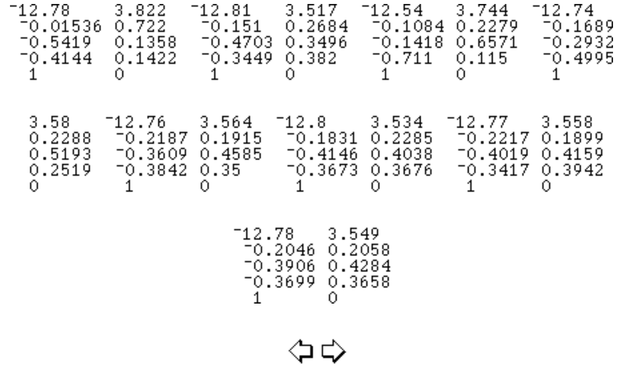
log Gap vs iteration #



log Infeasibility vs iteration #



```
     MATPRINT ΦTAB
  0 1 ¯10.82 ¯0.6931 ¯12.21 ¯0.9163 ¯11.98 ¯0.4055 4.605
  0 0  ¯1     1        0      0        0      0     1
  0 0   0     0       ¯1      1        0      0     0
  0 0   0     0        0      0       ¯1      1     0
  1 0   1     1        1      1        1      1     0

4.605 4.605 ¯7.962 3.507  ¯8.012     3.548  ¯12.66   3.631
0     0      0     0.3333 ¯0.0001506 0.2057  ¯0.3253 0.122
1     0      0     0.3333  0.0002598 0.3685  ¯0.3168 0.4878
0     1      0     0.3333  0.0003976 0.4258  ¯0.3223 0.3902
0     0      1     0       1         0        1      0

 ¯12.78    3.822  ¯12.81   3.517  ¯12.54   3.744  ¯12.74
  ¯0.01536 0.722   ¯0.151  0.2684  ¯0.1084 0.2279  ¯0.1689
  ¯0.5419  0.1358  ¯0.4703 0.3496  ¯0.1418 0.6571  ¯0.2932
  ¯0.4144  0.1422  ¯0.3449 0.382   ¯0.711  0.115   ¯0.4995
   1       0        1      0        1      0        1
```



```
¯12.78    3.822  ¯12.81   3.517  ¯12.54   3.744  ¯12.74
 ¯0.01536 0.722   ¯0.151  0.2684  ¯0.1084 0.2279  ¯0.1689
 ¯0.5419  0.1358  ¯0.4703 0.3496  ¯0.1418 0.6571  ¯0.2932
 ¯0.4144  0.1422  ¯0.3449 0.382   ¯0.711  0.115   ¯0.4995
  1       0        1      0        1      0        1

  3.58   ¯12.76   3.564 ¯12.8    3.534 ¯12.77   3.558
  0.2288  ¯0.2187 0.1915  ¯0.1831 0.2285  ¯0.2217 0.1899
  0.5193  ¯0.3609 0.4585  ¯0.4146 0.4038  ¯0.4019 0.4159
  0.2519  ¯0.3842 0.35    ¯0.3673 0.3676  ¯0.3417 0.3942
  0        1      0        1      0        1      0

                ¯12.78   3.549
                 ¯0.2046 0.2058
                 ¯0.3906 0.4284
                 ¯0.3699 0.3658
                  1      0
```

```
      ρTab_feas
5 11
      MATPRINT ⍕Tab_feas
  0 1 ¯12.44 ¯6.293 ¯13.09 ¯5.64 ¯12.98 ¯5.753 3.071 3.724
  0 0  ¯1     1      0      0      0      0     1     0
  0 0   0     0     ¯1      1      0      0     0     1
  0 0   0     0      0      0     ¯1      1     0     0
  1 0   1     1      1      1      1      1     0     0

                            3.611
                            0
                            0
                            1
                            0
```

Current Grid Points

```
 # t p |    1        2        3        4        5        6
 2 0 1 | 1.000000
 3 0 1 |          1.000000
 4 0 1 |                   1.000000
 5 0 1 |                            1.000000
 6 0 1 |                                     1.000000
 7 0 1 |                                              1.00000
 8 0 2 | 1.000000
 9 0 2 |          1.000000
10 0 2 |                   1.000000
11 0 1 | 0.166667 0.166667 0.166667 0.166667 0.166667 0.16666
12 0 2 | 0.333333 0.333333 0.333333
13 0 1 | 0.101935 0.101935 0.227933 0.227933 0.182347 0.15791
14 0 2 | 0.205997 0.368498 0.425505
```

(2) u_{11}	(3) u_{12}	(4) u_{13}	(5) u_{14}	(6) u_{15}
¯1.0819E1	¯6.9314E¯1	¯1.2206E1	¯9.1629E¯1	¯1.1982E1
¯1	1	0	0	0
0	0	¯1	1	¯1
0	0	0	0	0
1	1	1	1	1

(7) u_{16}	(8) u_{21}	(9) u_{22}	(10) u_{23}	(11) u_{17}	(12) u_{24}
¯4.0546E¯1	4.6051	4.6051	4.6051	¯7.9623	3.5065E0
0	1	0	0	0	3.3333E¯1
0	0	1	0	¯1.6666E¯1	3.3333E¯1
1	0	0	1	1.6666E¯1	3.3333E¯1
1	0	0	0	1	0

(13) u_{18}	(14) u_{25}	RHS
¯8.1550E0	3.5482E0	0
2.1908E¯8	2.0599E¯1	0
¯1.8234E¯1	3.6849E¯1	0
1.5791E¯1	4.2550E¯1	0
1	0	1

Initial LP

Iteration 1

LP Solution

Col	Posy	Type	Value
1	0	0	7.60090246
12	2	0	0.00000000
4	1	0	1.00000000
9	2	0	1.00000000
13	1	0	0.00000000

= ln 1999.9999 *(objective)*

```
Determinant of the basis matrix = 0.05263893905
```

$$-12.206\times\overset{u_{13}}{1} + 4.6051\times\overset{u_{22}}{1} + 3.5065\times\overset{u_{24}}{0} - 8.155\times\overset{u_{18}}{0} = 7.6009$$

$$\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\times1 + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\times1 + \begin{bmatrix}1/3\\1/3\\1/3\\0\end{bmatrix}\times0 + \begin{bmatrix}2.2\times10^{-8}\\-0.182\\0.158\\1\end{bmatrix}\times0 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Simplex Multipliers

$$\boxed{\pi = C_B^T\left[A^B\right]^{-1}}$$

$$\pi = \begin{bmatrix}-12.206\\4.6051\\3.5065\\-8.155\end{bmatrix}^T \begin{bmatrix}0&0&1/3&2.2\times10^{-8}\\-1&1&1/3&-0.182\\0&0&1/3&0.158\\1&0&0&1\end{bmatrix}^{-1}$$

$$= [\underbrace{\,4.10616,\ 4.60517,\ 1.80834}_{\textit{multipliers for orthogonality constraints}},\ -7.6009]$$

Exponentiating the Simplex Multipliers yields:

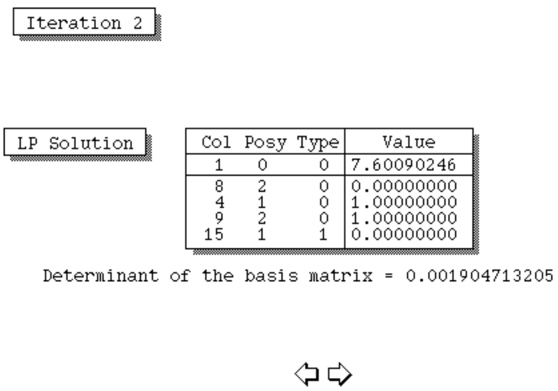
```
X = 60.71333761 100 6.100313127
```

$\leftarrow$ *order quantities*

```
Weights (ρ):
0.17142  0.025275  0.4163  0.052038  0.33304  0.0019047
0.36395 0.59947 0.036569

Objective functions:
  Primal: 4804.1          Dual: 2000
  Duality Gap: 2804.1        = 140.2 percent

Constraints:    | Value          | 1.6681
                | Infeasibility  | 0.6681
                | Lambda         | 1

Type-1 grid points (#15 16) added for posynomials 1 2
```

Added Grid Points:

```
 # t p |    1        2        3        4        5        6
15 1 1 | 0.171424 0.025276 0.416309 0.052039 0.333048 0.001905
16 1 2 | 0.363959 0.599471 0.036570
```

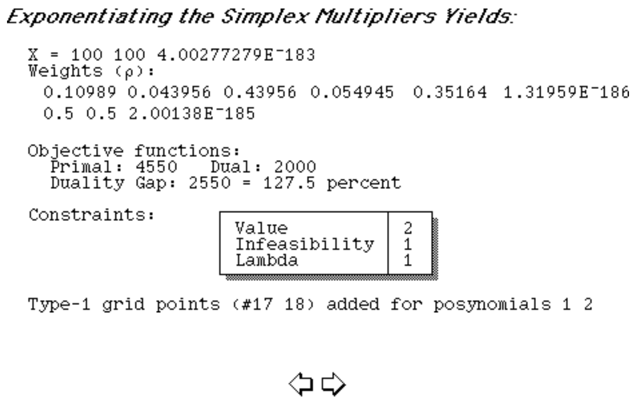


Iteration 2

LP Solution

Col	Posy	Type	Value
1	0	0	7.60090246
8	2	0	0.00000000
4	1	0	1.00000000
9	2	0	1.00000000
15	1	1	0.00000000

```
Determinant of the basis matrix = 0.001904713205
```

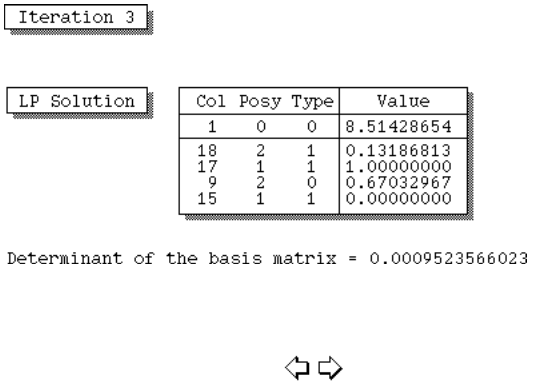


Exponentiating the Simplex Multipliers Yields:

```
X = 100 100 4.00277279E¯183
Weights (ρ):
  0.10989 0.043956 0.43956 0.054945  0.35164  1.31959E¯186
  0.5 0.5 2.00138E¯185

Objective functions:
   Primal: 4550   Dual: 2000
   Duality Gap: 2550 = 127.5 percent
Constraints:
```

Value	2
Infeasibility	1
Lambda	1

```
Type-1 grid points (#17 18) added for posynomials 1 2
```

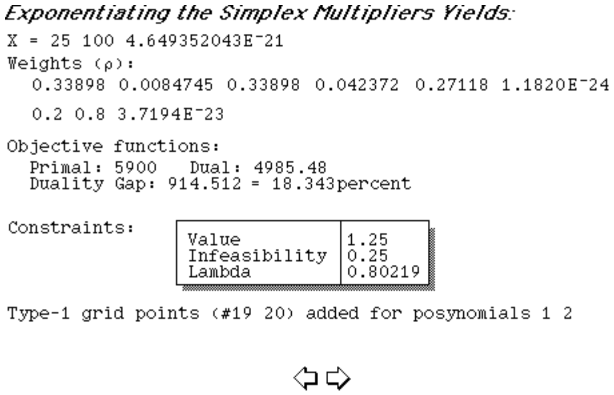


Iteration 3

LP Solution

Col	Posy	Type	Value
1	0	0	8.51428654
18	2	1	0.13186813
17	1	1	1.00000000
9	2	0	0.67032967
15	1	1	0.00000000

```
Determinant of the basis matrix = 0.0009523566023
```

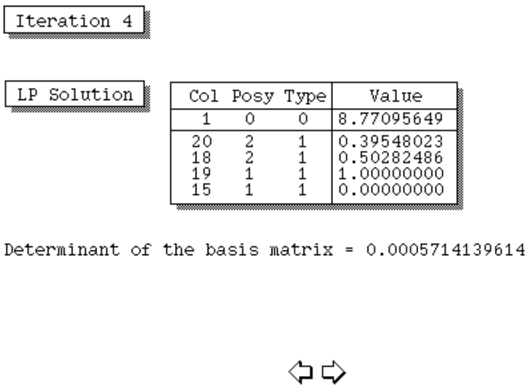


Exponentiating the Simplex Multipliers Yields:

```
X = 25 100 4.649352043E¯21
Weights (ρ):
  0.33898  0.0084745  0.33898  0.042372  0.27118  1.1820E¯24
  0.2  0.8  3.7194E¯23
Objective functions:
   Primal: 5900   Dual: 4985.48
   Duality Gap: 914.512 = 18.343percent

Constraints:
```

Value	1.25
Infeasibility	0.25
Lambda	0.80219

```
Type-1 grid points (#19 20) added for posynomials 1 2
```

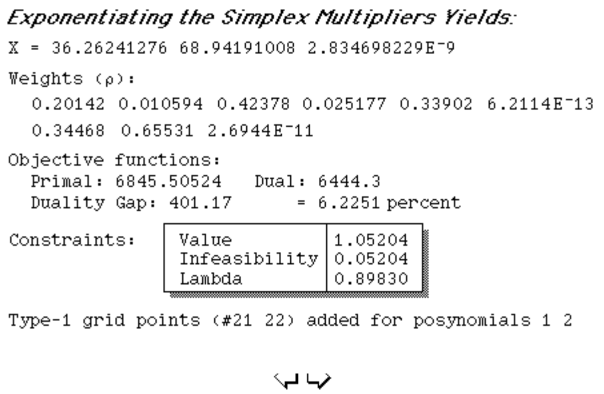


Iteration 4

LP Solution

Col	Posy	Type	Value
1	0	0	8.77095649
20	2	1	0.39548023
18	2	1	0.50282486
19	1	1	1.00000000
15	1	1	0.00000000

```
Determinant of the basis matrix = 0.0005714139614
```

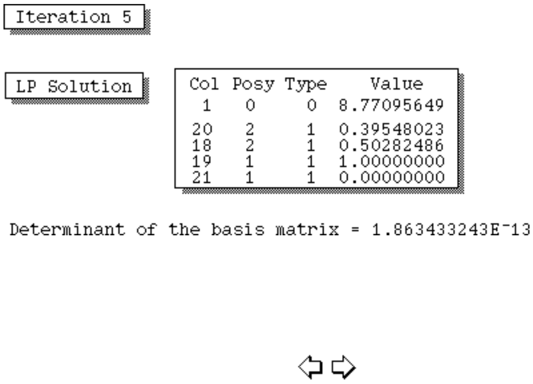


Exponentiating the Simplex Multipliers Yields:

```
X = 36.26241276 68.94191008 2.834698229E¯9
Weights (ρ):
  0.20142  0.010594  0.42378  0.025177  0.33902  6.2114E¯13
  0.34468  0.65531  2.6944E¯11
Objective functions:
   Primal: 6845.50524   Dual: 6444.3
   Duality Gap: 401.17      = 6.2251 percent
Constraints:
```

Value	1.05204
Infeasibility	0.05204
Lambda	0.89830

```
Type-1 grid points (#21 22) added for posynomials 1 2
```

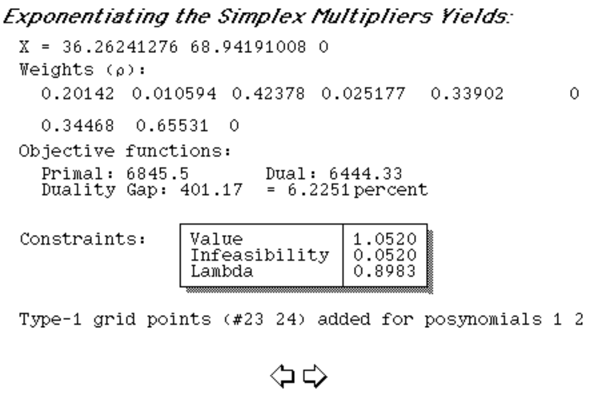


Iteration 5

LP Solution

Col	Posy	Type	Value
1	0	0	8.77095649
20	2	1	0.39548023
18	2	1	0.50282486
19	1	1	1.00000000
21	1	1	0.00000000

```
Determinant of the basis matrix = 1.863433243E¯13
```



Exponentiating the Simplex Multipliers Yields:

```
X = 36.26241276 68.94191008 0
Weights (ρ):
  0.20142  0.010594  0.42378  0.025177   0.33902       0
  0.34468  0.65531  0
Objective functions:
   Primal: 6845.5       Dual: 6444.33
   Duality Gap: 401.17  = 6.2251percent

Constraints:
```

Value	1.0520
Infeasibility	0.0520
Lambda	0.8983

```
Type-1 grid points (#23 24) added for posynomials 1 2
```