DANTZIG - WOLFE DECOMPOSITION

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- ☞ Representation Theorem
- ☞ Reformulating problem as optimizing combination of extreme pts & rays
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Consider the Linear Programming Problem

$$\begin{aligned} &\text{Maximize } c^T x \\ &\text{subject to } Ax \le b \\ &\qquad\qquad x \in X \end{aligned}$$

where X is a polyhedral set,
e.g., the solution set of a linear system of equations and/or inequalities such as

$$X = \{x : Dx \le e, x \ge 0\}$$

$$\begin{aligned} &\text{Maximize } c^T x \\ &\text{subject to } Ax \le b \\ &\qquad\qquad x \in X \end{aligned}$$

In most practical applications of decomposition,

the constraint $x \in X$ consists of the easy-to-handle constraints, and

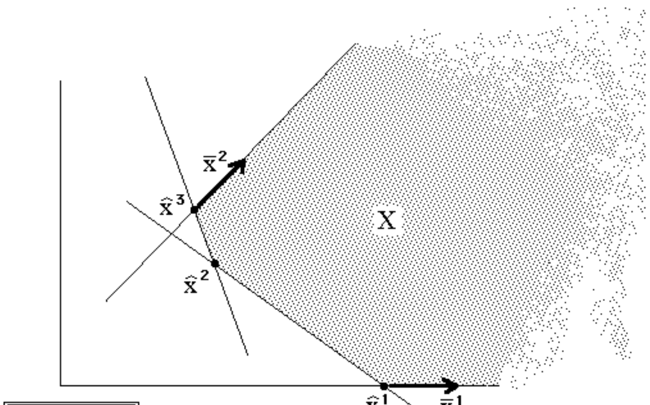
the constraint $Ax \le b$ consists of the complicating constraints.

If X is polyhedral, then every point in can be expressed as a linear combination of its

extreme points: $\hat{x}^i,\ i = 1,2,\cdots I$

and

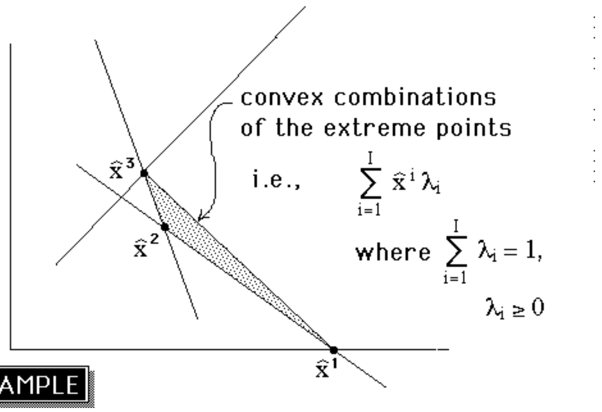
extreme rays: $\bar{x}^j,\ j = 1,2,\cdots J$



EXAMPLE

If X is polyhedral, then every point in can be expressed as a linear combination of its extreme points and extreme rays.

$$x \in X \iff \begin{cases} \exists\ \lambda_i \ge 0 \text{ where } \sum_{i=1}^{I} \lambda_i = 1, \\ \&\ \mu_j \ge 0 \\ \text{such that} \\ x = \sum_{i=1}^{I} \hat{x}^i \lambda_i + \sum_{j=1}^{J} \bar{x}^j \mu_j \end{cases}$$



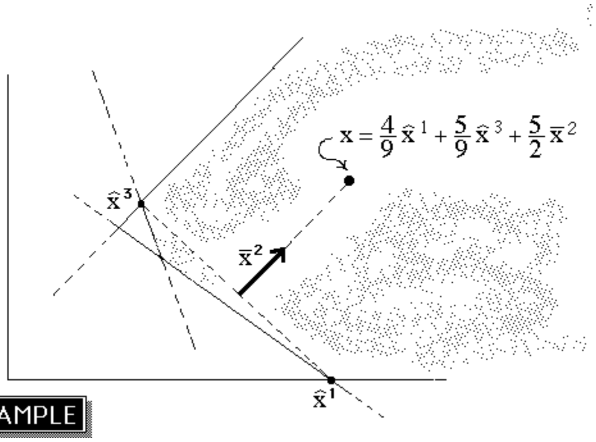
convex combinations of the extreme points

i.e., $\sum_{i=1}^{I} \hat{x}^i \lambda_i$

where $\sum_{i=1}^{I} \lambda_i = 1,$

$\lambda_i \ge 0$

EXAMPLE



EXAMPLE

Substituting the linear combination of extreme points & rays into the LP, we get

$$\text{Maximize } c^T\left(\sum_{i=1}^{I}\hat{x}^i\lambda_i + \sum_{j=1}^{J}\bar{x}^j\mu_j\right)$$

$$\text{subject to } \begin{cases} A\left(\sum_{i=1}^{I}\hat{x}^i\lambda_i + \sum_{j=1}^{J}\bar{x}^j\mu_j\right) \le b \\ \sum_{i=1}^{I}\lambda_i = 1 \\ \lambda_i \ge 0,\ \mu_j \ge 0 \end{cases}$$

LP in the variables λ_i *and* μ_j

$$\text{Maximize } \sum_{i=1}^{I} c^T\hat{x}^i\lambda_i + \sum_{j=1}^{J} c^T\bar{x}^j\mu_j$$

$$\text{subject to } \begin{cases} \sum_{i=1}^{I} A\hat{x}^i\lambda_i + \sum_{j=1}^{J} A\bar{x}^j\mu_j \le b \\ \sum_{i=1}^{I}\lambda_i = 1 \\ \lambda_i \ge 0,\ \mu_j \ge 0 \end{cases}$$

That is,

$$\hat{f}^i \equiv c^T\hat{x}^i \qquad \bar{f}^j \equiv c^T\bar{x}^j \qquad \hat{p}^i \equiv A\hat{x}^i \qquad \bar{p}^j \equiv A\bar{x}^j$$

$$\text{Maximize } \sum_{i=1}^{I}\hat{f}^i\lambda_i + \sum_{j=1}^{J}\bar{f}^j\mu_j$$

$$\text{subject to } \begin{cases} \sum_{i=1}^{I}\hat{p}^i\lambda_i + \sum_{j=1}^{J}\bar{p}^j\mu_j \le b \\ \sum_{i=1}^{I}\lambda_i = 1 \\ \lambda_i \ge 0,\ \mu_j \ge 0 \end{cases}$$

Thus, by a change of variables, our original LP

$$\boxed{\begin{array}{l} \text{Maximize } c^T x \\ \text{subject to } Ax \le b \\ \quad x \in X \end{array}}$$

has been transformed into the LP

$$\boxed{\begin{array}{l} \text{Maximize } \sum_i \hat{f}^i\lambda_i + \sum_j \bar{f}^j\mu_j \\ \text{subject to } \sum_i \hat{p}^i\lambda_i + \sum_j \bar{p}^j\mu_j \le b \\ \qquad \sum_i \lambda_i = 1 \\ \qquad \lambda_i \ge 0,\ \mu_j \ge 0 \end{array}}$$

Suppose that A has m' rows and that m" rows define X

$$\boxed{\begin{array}{l} \text{Maximize } c^T x \\ \text{subject to } Ax \le b \\ \quad x \in X \end{array}} \quad \text{m'+m" rows}$$

$$\boxed{\begin{array}{l} \text{Maximize } \sum_i \hat{f}^i\lambda_i + \sum_j \bar{f}^j\mu_j \\ \text{subject to } \sum_i \hat{p}^i\lambda_i + \sum_j \bar{p}^j\mu_j \le b \\ \qquad \sum_i \lambda_i = 1 \\ \qquad \lambda_i \ge 0,\ \mu_j \ge 0 \end{array}}$$

only m'+1 rows

but $\underbrace{I + J}_{\text{usually an "astronomical" number}}$ columns

Even though our new LP formulation may have a huge number of variables, we know that the optimal (basic) solution will have at most m'+1 nonzero variables!

The Dantzig-Wolfe method does NOT compute and store all I+J columns of the LP, but will generate only a very small subset of the more "attractive" columns.

Suppose that the Revised Simplex Method is being used to solve the master problem, and that the current simplex multiplier vector is

$$\pi = [\underbrace{\omega}_{\text{m' elements}},\ \underbrace{\alpha}_{\text{multiplier for convexity row}}]$$

What are the relative profits of the nonbasic columns $\begin{bmatrix}\hat{p}^i \\ 1\end{bmatrix}$ & $\begin{bmatrix}\bar{p}^j \\ 0\end{bmatrix}$?

Reduced Costs

$$\hat{f}^i - [\omega, \alpha]\begin{bmatrix}\hat{p}^i \\ 1\end{bmatrix} = \hat{f}^i - \omega\hat{p}^i - \alpha$$

$$= c^T\hat{x}^i - \omega A\hat{x}^i - \alpha$$

$$= \left[c^T - \omega A\right]\hat{x}^i - \alpha$$

$$\bar{f}^j - [\omega, \alpha]\begin{bmatrix}\bar{p}^j \\ 0\end{bmatrix} = \bar{f}^j - \omega\,\bar{p}^j$$

$$= c^T\bar{x}^j - \omega A\bar{x}^j$$

$$= \left[c^T - \omega A\right]\bar{x}^j$$

Suppose that we select the column with the *greatest relative profit* to enter into the basis....

This column can be determined by solving the subproblem

$$\boxed{\begin{array}{l} \text{Maximize } \left[c^T - \omega A\right]x \\ \text{subject to } \quad x \in X \end{array}}$$

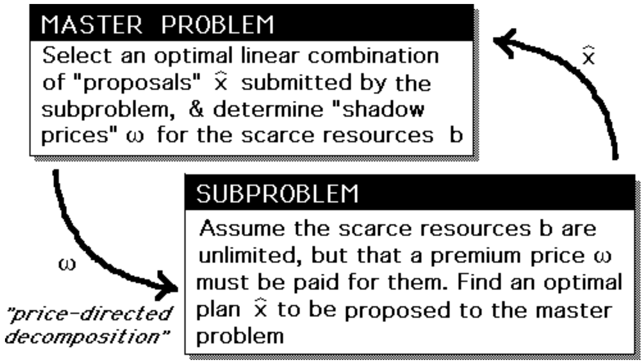
which is an LP problem whose solution is an extreme point $\hat{x}^i$ if it is bounded!

subproblem

$$\boxed{\begin{array}{l} \text{Maximize } \left[c^T - \omega A\right]x \\ \text{subject to } \quad x \in X \end{array}}$$

Therefore, we can apply the simplex method to the subproblem, and obtain an extreme point if the solution is bounded.

If the solution of the subproblem is unbounded, then the simplex method will give us a ray along which the LP is unbounded!



MASTER PROBLEM

Select an optimal linear combination of "proposals" $\hat{x}$ submitted by the subproblem, & determine "shadow prices" ω for the scarce resources b

SUBPROBLEM

Assume the scarce resources b are unlimited, but that a premium price ω must be paid for them. Find an optimal plan $\hat{x}$ to be proposed to the master problem

"price-directed decomposition"

subproblem

$$\boxed{\begin{array}{l} \text{Maximize } \left[c^T - \omega A\right]x \\ \text{subject to } \quad x \in X \end{array}}$$

In order for this scheme to be efficient, the subproblem must be solved very efficiently.... For example, the constraint $x \in X$ might be

- the set of assignment problem constraints
- the set of transportation problem constraints
- other network flow constraints
- separable, i.e., X is the "cartesian product" of several independent sets.

EXAMPLE

$$\boxed{\begin{array}{ll} \text{Maximize} & x_1 + 2x_2 + x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \le 12 \\ & -x_1 + x_2 \quad \le 2 \\ & -x_1 + 2x_2 \quad \le 8 \\ & \qquad x_3 \le 3 \\ & x_j \ge 0,\ j=1,2,3 \end{array}}$$

Denote

$$c = \begin{bmatrix}1\\2\\1\end{bmatrix},\ A = [\,1\ \ 1\ \ 1\,],\ b = [12]$$

Then the LP could be restated as

$$\boxed{\begin{array}{l} \text{Maximize } c^T x \\ \text{subject to } Ax \le b \\ \qquad\qquad x \in X \end{array}}$$

where

$$X = \{x : Dx \le e, x \ge 0\}$$

$$\& \quad D = \begin{bmatrix}-1 & 1 & 0\\ -1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix},\ e = \begin{bmatrix}2\\8\\3\end{bmatrix}$$

$$X = \{x : Dx \le e, x \ge 0\}$$

$$\& \quad D = \left[\begin{array}{cc:c}-1 & 1 & 0\\ -1 & 2 & 0\\ \hdashline 0 & 0 & 1\end{array}\right],\ e = \begin{bmatrix}2\\8\\3\end{bmatrix}$$

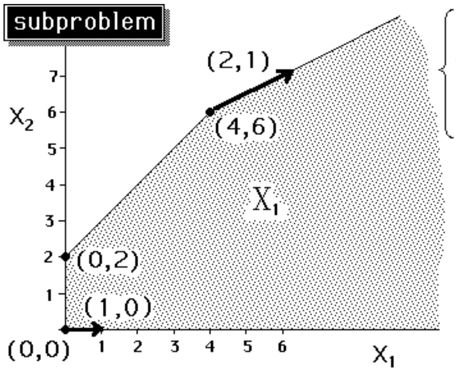
In this example, the matrix D has a block-diagonal structure, so that the problem is separable

That is, $X = X_1 \times X_2$ *(Cartesian product)*

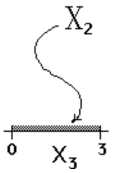
where $X_1 = \left\{(x_1,x_2) : \begin{bmatrix}-1 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \le \begin{bmatrix}2\\8\end{bmatrix}, \begin{bmatrix}x_1\\x_2\end{bmatrix} \ge 0\right\}$

and $X_2 = \{x_3 : 0 \le x_3 \le 3\}$

subproblem



$$\begin{cases} -x_1 + x_2 \le 2 \\ -x_1 + 2x_2 \le 8 \\ x_3 \le 3 \\ x_j \ge 0,\ j=1,2,3 \end{cases}$$



> *In first solving this example by Dantzig-Wolfe Decomposition, however, we will not make the most efficient use of this separability. Later, we will return to this example and make full use of the separability!*

Extreme Pts

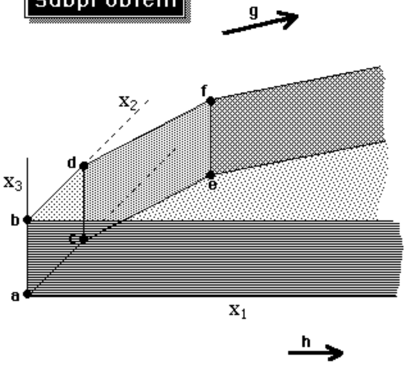
$$\begin{array}{ccc} a & b & c \\ \begin{bmatrix}0\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\3\end{bmatrix}, & \begin{bmatrix}0\\2\\0\end{bmatrix}, \end{array}$$

$$\begin{array}{ccc} d & e & f \\ \begin{bmatrix}0\\2\\3\end{bmatrix}, & \begin{bmatrix}4\\6\\0\end{bmatrix}, & \begin{bmatrix}4\\6\\3\end{bmatrix} \end{array}$$

Extreme Rays

$$\begin{array}{cc} g & h \\ \begin{bmatrix}2\\1\\0\end{bmatrix}, & \begin{bmatrix}1\\0\\0\end{bmatrix} \end{array}$$

subproblem



	a	b	c	d	e	f	g	h
"proposals"	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\3\end{bmatrix}$	$\begin{bmatrix}0\\2\\0\end{bmatrix}$	$\begin{bmatrix}0\\2\\3\end{bmatrix}$	$\begin{bmatrix}4\\6\\0\end{bmatrix}$	$\begin{bmatrix}4\\6\\3\end{bmatrix}$	$\begin{bmatrix}2\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$
profit $c^T\hat{x} = [1,\ 2,\ 1]\hat{x}$ $= \hat{x}_1 + 2\hat{x}_2 + \hat{x}_3$	0	3	4	7	16	19	4	1
scarce resource usage $A\hat{x} = [1, 1, 1]\hat{x}$ $= \hat{x}_1 + \hat{x}_2 + \hat{x}_3$	0	3	2	5	10	13	3	1

Master Problem

-z	ext.pt.proposals						ext.ray proposals		slack	rhs	
1	0	3	4	7	16	19	4	1	0	0	max
0	0	3	2	5	10	13	3	1	1	12	
0	1	1	1	1	1	1	0	0	0	1	

This is the *complete* master problem, which includes every possible proposal from the subproblem.... seldom, however, is the master problem solved with all proposals!

Let's begin with a single proposal from the subproblem, namely $\hat{x}^1 = [0,0,0]^T$.

"do nothing"

partial master problem

-z	λ_1	s	rhs
1	0	0	0
0	0	1	12
0	1	0	1

The "optimal" solution is the unique feasible solution, namely $-z=0$, $\lambda_1=1$, and $s=12$ with basis B=[1,3,2]

The Master decision-maker chooses the only proposal available to him, leaving 12 units of unused resource.

Basis matrix & its inverse

$$\tilde{A}^B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\tilde{A}^B]^{-1}$$

Simplex multipliers

$$\pi = [\omega, \alpha] = \tilde{c}_B\,[\tilde{A}^B]^{-1} = [0\ ,0]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= [0\ ,0]$$

Since the master decision-maker has a surplus of the resource, it has no value!

The subproblem decision-maker must now maximize his profit, considering the resource to be a free commodity with unlimited supply.

subproblem

$$\begin{array}{l} \text{Maximize } [c^T - \omega A]\,x \\ \text{subject to } \quad x \in X \end{array}$$

currently $\omega=0$

relative profit = profit

$$\text{Maximize } \ x_1 + 2x_2 + x_3$$

$$\text{subject to } \begin{cases} -x_1 + x_2 \le 2 \\ -x_1 + 2x_2 \le 8 \\ x_3 \le 3 \\ x_j \ge 0,\ j=1,2,3 \end{cases}$$

subproblem

In this example, the subproblem separates into two independent problems

$$\text{Maximize } x_1+2x_2+x_3 \quad \text{subject to} \begin{cases} -x_1+x_2 \le 2 \\ -x_1+2x_2 \le 8 \\ x_3 \le 3 \\ x_j \ge 0,\ j=1,2,3 \end{cases}$$

$$\text{Maximize } x_1+2x_2 \quad \text{subject to} \begin{cases} -x_1+x_2 \le 2 \\ -x_1+2x_2 \le 8 \\ x_j \ge 0,\ j=1,2 \end{cases}$$

$$\text{Maximize } x_3 \quad \text{subject to } 0 \le x_3 \le 3$$

$$\text{Maximize } x_1+2x_2 \quad \text{subject to} \begin{cases} -x_1+x_2 \le 2 \\ -x_1+2x_2 \le 8 \\ x_j \ge 0,\ j=1,2 \end{cases}$$

-z	x_1	x_2	x_4	x_5	
1	1	2	0	0	0
0	-1	1	1	0	2
0	-1	2	0	1	8

initial tableau

-z	x_1	x_2	x_4	x_5	
1	0	0	4	-3	-16
0	0	1	-1	1	6
0	1	0	-2	1	4

↑ pivot

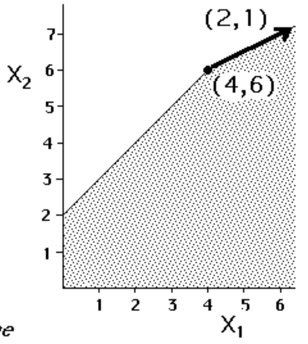
column chosen for pivot has no positive element on which to pivot!

This condition means that the LP solution is unbounded!

-z	x_1	x_2	x_4	x_5	
1	0	0	4	-3	-16
0	0	1	-1	1	6
0	1	0	-2	1	4

$$\Rightarrow \begin{cases} z = 16+4x_4 \\ x_2 = 6+x_4 \\ x_1 = 4+2x_4 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}_{\textit{extreme point}} + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\textit{extreme ray}} x_4$$



The solution to the problem in x_3 is bounded ($x_3 = 3$), but because the problem in x_1 & x_2 is unbounded, the subproblem in all three variables x_1, x_2, & x_3 is unbounded, along the ray

$$\text{Maximize } x_3 \quad \text{subject to } 0 \le x_3 \le 3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \leftarrow \textit{i.e., } x_3 \textit{ does not change along this ray!}$$

$$\text{relative profit } = x_1+2x_2+x_3 = 4 > 0$$

	proposal $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
profit $c^T\bar{x} = [\,1,\ 2,\ 1\,]\,\bar{x} = \bar{x}_1+2\bar{x}_2+\bar{x}_3$	4
scarce resource usage $A\bar{x} = [1,\ 1,\ 1]\,\bar{x} = \bar{x}_1+\bar{x}_2+\bar{x}_3$	3

The column to be added to the master problem is therefore

$$\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{profit} \\ \leftarrow \text{resource usage} \\ \leftarrow \text{convexity row} \end{array}$$

Generating Column for Master Problem

Master Problem

-z	λ_1	s	μ_1	rhs
1	0	0	4	0
0	0	1	3	12
0	1	0	0	1

The optimal solution to this new master problem is $\mu_1 = 4,\ \lambda_1 = 1,\ z = 16,\ s = 0$

with optimal basis B=[4,2]

$$\tilde{A}^B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},\ [\tilde{A}^B]^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$$

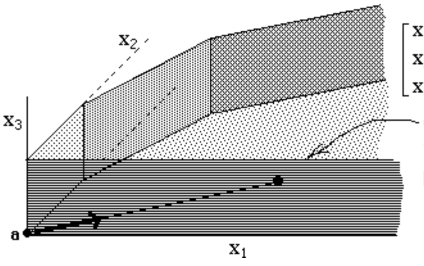
$$\pi = \tilde{c}_B\,[\tilde{A}^B]^{-1} = [\ 4\ ,\ 0\,] \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = [\ 4/3\ ,\ 0\ ]$$

$$\left.\begin{array}{l} \mu_1 = 4 \\ \lambda_1 = 1 \end{array}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \mu_1 = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

That is, the master decision-maker decides to combine the first proposal [0,0,0] with 4 times the second proposal [2, 1, 0], yielding the feasible solution x_1=8, x_2=4, x_3=0, which results in no slack in the resource usage (s=0).

$$\pi = [\omega,\ \alpha] = [\ 4/3\ ,\ 0\ ]$$

The "shadow price" of the resource is now 4/3



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overset{\text{extreme point}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} 1 + \overset{\text{extreme ray}}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}} 4$$

optimal solution of the partial master problem

subproblem

$$\begin{array}{ll}\text{Maximize} & \left[c^T - \omega A\right]x \\ \text{subject to} & x \in X\end{array}$$

currently $\omega = 4/3$

That is, the subproblem decision-maker must now maximize his "relative profit", which is the profit of the activities x_1, x_2, and x_3 *minus* the value of the resource which they use.

RELATIVE PROFIT

$$\text{Maximize} \quad \underbrace{x_1 + 2x_2 + x_3}_{} - 4/3 \; \underbrace{(x_1 + x_2 + x_3)}_{}$$

$$\textit{profit} - \begin{bmatrix}\textit{value of} \\ \textit{resource}\end{bmatrix} \times \begin{bmatrix}\textit{amt. of resource} \\ \textit{used}\end{bmatrix}$$

activities 1 and 3 require quantities of the resource which exceed their profits, yielding negative relative profits!

$$\text{Maximize} \; -1/3x_1 + 2/3x_2 - 1/3\,x_3$$

subproblem

subproblem

Again, the subproblem may be separated into two independent LPs

$$\begin{array}{l}\text{Maximize} \; -1/3x_1 + 2/3x_2 - 1/3\,x_3 \\ \text{subject to} \begin{cases} -x_1 + x_2 \le 2 \\ -x_1 + 2x_2 \le 8 \\ x_3 \le 3 \\ x_j \ge 0,\ j=1,2,3 \end{cases}\end{array}$$

$$\begin{array}{l}\text{Maximize} \; -1/3x_1 + 2/3x_2 \\ \text{subject to} \begin{cases} -x_1 + x_2 \le 2 \\ -x_1 + 2x_2 \le 8 \\ x_j \ge 0,\ j=1,2 \end{cases}\end{array} \qquad \begin{array}{l}\text{Maximize} \; -1/3\,x_3 \\ \text{subject to} \\ \quad 0 \le x_3 \le 3\end{array}$$

subproblem

$$\begin{array}{l}\text{Maximize} \; -1/3x_1 + 2/3x_2 \\ \text{subject to} \begin{cases} -x_1 + x_2 \le 2 \\ -x_1 + 2x_2 \le 8 \\ x_j \ge 0,\ j=1,2 \end{cases}\end{array}$$

optimum

$x_1 = 4$ & $x_2 = 6$

profit $z' = 8/3$

$$\begin{array}{l}\text{Maximize} \; -1/3\,x_3 \\ \text{subject to} \\ \quad 0 \le x_3 \le 3\end{array}$$

optimum $x_3 = 0$

profit $z'' = 0$

relative profit

$$= 8/3 + 0 - \underset{\alpha}{0} > 0$$

proposal $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\6\\0\end{bmatrix}$

	proposal $\begin{bmatrix}4\\6\\0\end{bmatrix}$
profit $c^T\hat{x} = [1,\ 2,\ 1]\hat{x}$ $= \hat{x}_1 + 2\hat{x}_2 + \hat{x}_3$	16
scarce resource usage $A\hat{x} = [1,\ 1,\ 1]\hat{x}$ $= \hat{x}_1 + \hat{x}_2 + \hat{x}_3$	10

The column to be added to the master problem is therefore

$$\begin{bmatrix}16\\10\\1\end{bmatrix}\begin{array}{l}\leftarrow \text{profit}\\ \leftarrow \text{resource usage}\\ \leftarrow \text{convexity row}\end{array}$$

Generating Column for Master Problem

partial master problem

-z	λ_1	s	μ_1	λ_2	rhs
1	0	0	4	16	0
0	0	1	3	10	12
0	1	0	0	1	1

We add the column for this new proposal from the subproblem, and re-solve the master problem, obtaining the optimal solution

$$\mu_1 = 2/3 \;\&\; \lambda_2 = 1$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\lambda_1 + \begin{bmatrix}4\\6\\0\end{bmatrix}\lambda_2 + \begin{bmatrix}2\\1\\0\end{bmatrix}\mu_1$$

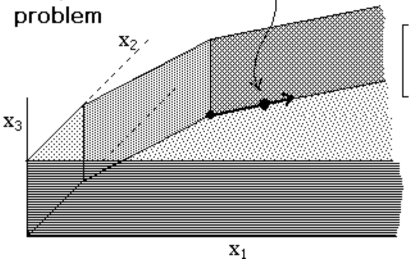
The master problem decision-maker combines two subproblem proposals so as to maximize his profit:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\6\\0\end{bmatrix}\times 1 + \begin{bmatrix}2\\1\\0\end{bmatrix}\times 2/3 = \begin{bmatrix}16/3\\20/3\\0\end{bmatrix}$$

$$\pi = [\omega, \alpha] = [4, 16]\begin{bmatrix}3 & 10\\0 & 1\end{bmatrix}^{-1} = [4, 16]\begin{bmatrix}1/3 & -10/3\\0 & 1\end{bmatrix}$$

$$= [4/3\,,\, 8/3]$$

optimal solution of the partial master problem



$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \overset{\text{extreme point}}{\begin{bmatrix}4\\6\\0\end{bmatrix}}1 + \overset{\text{extreme ray}}{\begin{bmatrix}2\\1\\0\end{bmatrix}}2/3$$

RELATIVE PROFIT

$$\text{Maximize}\quad \underbrace{x_1 + 2x_2 + x_3}\ -\ \tfrac{4}{3}\ \underbrace{(x_1 + x_2 + x_3)}\ \ - \overset{8/3}{\alpha}$$

$$\text{profit} - \begin{bmatrix}\text{value of}\\ \text{resource}\end{bmatrix} \times \begin{bmatrix}\text{amt. of resource}\\ \text{used}\end{bmatrix}$$

$$\text{Maximize}\quad -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3$$

subproblem

The objective is unchanged from the previous iteration, and so the optimal solution of the subproblem decision-maker is still $x = \begin{bmatrix}4\\6\\0\end{bmatrix}$ but relative profit is now

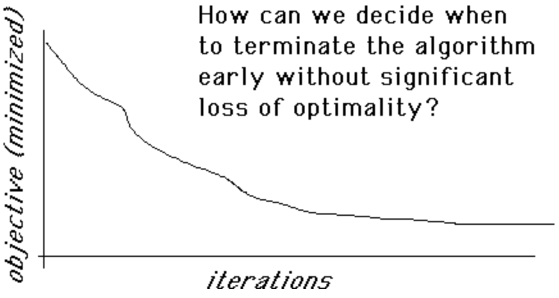
$$\tfrac{8}{3} + 0 - \underset{\alpha}{\tfrac{8}{3}} = 0$$

That is, the subproblem is unable to find a proposal whose master-problem column would have a positive relative profit!

Therefore, the solution to the latest partial master problem would be optimal, even if every possible column were added to obtain the complete master problem.

In theory, the Dantzig-Wolfe Decomposition algorithm converges in a *finite* number of iterations (since X, if polyhedral, has a finite number of extreme points and extreme rays).

But *in practice*, early iterations produce substantial improvement of the objective, while improvements become smaller as the algorithm progresses...



How can we decide when to terminate the algorithm early without significant loss of optimality?

Error Bounds If we stop while the subproblem is still producing proposals with (perhaps small) relative profits, what will be the error? That is, how far from optimal will the current master problem solution be?

Assume that the feasible region is bounded, i.e., we can neglect extreme rays.

Suppose that we have solved the current

partial master problem

$$\text{Maximize}\ \sum_i \hat{f}^i \lambda_i$$

$$\text{subject to}\ \begin{cases} \sum_i \hat{p}^i \lambda_i \le b \\ \sum_i \lambda_i = 1 \\ \lambda_i \ge 0 \end{cases}$$

Current simplex multiplier vector $\pi = [\omega, \alpha]$

multiply each resource constraint by its "shadow price" and add

$$\begin{cases} \omega \sum_i \hat{p}^i \lambda_i + \omega I s = \omega b \\ \alpha \sum_i \lambda_i = \alpha 1 \end{cases}$$

Add above 2 constraints $\omega \sum_i \hat{p}^i \lambda_i + \omega I s + \alpha \sum_i \lambda_i = \omega b + \alpha$

Rearrange terms $\sum_i (\omega \hat{p}^i + \alpha)\lambda_i + \omega s = \omega b + \alpha$

$$\begin{cases} z = \sum_i \hat{f}^i \lambda_i \\ \omega b + \alpha = \sum_i (\omega \hat{p}^i + \alpha)\lambda_i + \omega s \end{cases}$$

Subtract the second equation from the first:

$$z - [\omega b + \alpha] + \underset{\text{nonnegative}}{\omega s} = \sum_i \left[\hat{f}^i - (\omega \hat{p}^i + \alpha)\right]\lambda_i$$

$$\Rightarrow\quad \underset{\text{current value of LP objective}}{z - [\omega b + \alpha]} \le \sum_i \underset{\text{relative profits}}{\left[\hat{f}^i - (\omega \hat{p}^i + \alpha)\right]}\lambda_i$$

Suppose that we denote the maximum relative profit found at the current iteration by $\hat{\mathbf{f}}^{\mathbf{max}}$.

$$z - \left[\omega\,\mathbf{b} + \alpha\right] \le \sum_i \underbrace{\left[\hat{\mathbf{f}}^i - \left(\omega\,\hat{\mathbf{p}}^i + \alpha\right)\right]}_{\textit{relative profits}} \lambda_i \;\le\; \sum_i \hat{\mathbf{f}}^{\mathbf{max}}\,\lambda_i \;=\; \hat{\mathbf{f}}^{\mathbf{max}}$$

That is, the gap between the current objective $\left[\omega\,\mathbf{b} + \alpha\right]$ and the maximum achievable profit cannot exceed $\hat{\mathbf{f}}^{\mathbf{max}}$!

Therefore, if we terminate the column-generation algorithm at the current iteration, we are guaranteed that we are within the value $\hat{\mathbf{f}}^{\mathbf{max}}$ *of the optimum!*