(s,S) inventory replenishment system

Example: (s,S) Inventory Control System with Periodic Review

Suppose that the inventory of a certain item is counted at the end of each business day, after a random demand has occurred.

If the number of units is ≤ s=2, the *reorder point*, enough is ordered to bring the inventory level up to S=6. (Assume that the replenishment is instantaneous.)

The daily *demand* has probability distribution:

Demand d	0	1	2	3	4	5	≥6
P{D=d}	0.04979	0.1494	0.224	0.224	0.168	0.1008	0.05041

(which is *Poisson* with expected value 3).

Define

- the *state* of the system to be the inventory level, i.e., I= {0, 1, 2, …6} *before* any replenishment, and
- the *stages* to be the days.

Transition probabilities are:

$$p_{ij} = \begin{cases} P\{D = i - j\} & \text{if i>s and i} \geq \text{j} \\ P\{D = S - j\} & \text{if i} \leq \text{s} \\ 0 \text{ otherwise} \end{cases}$$

Examples of system characteristics that may be of interest:

- if the system begins in state S=6, what is the probability of the first stockout occurring 3 days hence?

  ⓘ *Answer:* $f_{S,0}^{(3)}$

- if the system begins in state S=6, what is the average length of time until a stockout (zero inventory) occurs?

  ⓘ *Answer:* $m_{S,0}$

- if the system begins in state S, what is the expected number of stockouts during the next n days?

  ⓘ *Answer:* $\sum_{k \leq n} p_{S,j}^{(n)}$

- over a sufficiently long period of time, what is the fraction of the days that the end-of-day inventory level is i?

  ⓘ *Answer:* π_i

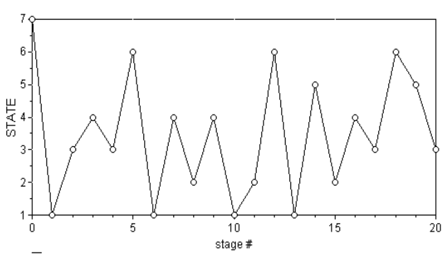
Transition Probabilities

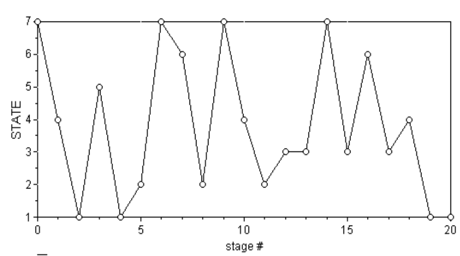
from \to	0	1	2	3	4	5	6
0	0.08392	0.1008	0.168	0.224	0.224	0.1494	0.04979
1	0.08392	0.1008	0.168	0.224	0.224	0.1494	0.04979
2	0.08392	0.1008	0.168	0.224	0.224	0.1494	0.04979
3	0.5768	0.224	0.1494	0.04979	0	0	0
4	0.3528	0.224	0.224	0.1494	0.04979	0	0
5	0.1847	0.168	0.224	0.224	0.1494	0.04979	0
6	0.08392	0.1008	0.168	0.224	0.224	0.1494	0.04979

Note that because the state is the inventory level before replenishments, the transition probabilities out of states 0, 1, and 2=s are identical to those for state S=6.

Thus, p_{64} = P{Demand = 6 – 4 = 2} = 0.224, and
p_{30} = P{Demand ≥ 3} = 0.5768

Two simulations of 20 stages of the system, beginning with full inventory:





In both of these realizations of the process, four stockouts have occurred during the first 20 days of operation.

The expected number of visits to state j during the first n stages, if the system begins in state i, may be found by summing the first n powers of P. In this particular case,

$$\sum_{k=1}^{20} P^k =$$

from \to	0	1	2	3	4	5	6
0	4.35285	2.92231	3.55777	3.65758	3.06261	1.85714	0.589734
1	4.35285	2.92231	3.55777	3.65758	3.06261	1.85714	0.589734
2	4.35285	2.92231	3.55777	3.65758	3.06261	1.85714	0.589734
3	4.72536	3.00256	3.52746	3.51926	2.90429	1.7616	0.559473
4	4.54813	3.01517	3.60129	3.6023	2.9318	1.74659	0.554716
5	4.43127	2.97713	3.60575	3.66134	3.00358	1.77409	0.546833
6	4.35285	2.92231	3.55777	3.65758	3.06261	1.85714	0.589734

Thus, we would expect approximately $\sum_{k=1}^{20} p_{6,0}^{(k)} = 4.35$ stockouts during the first twenty days of operation, if the inventory is initially full.

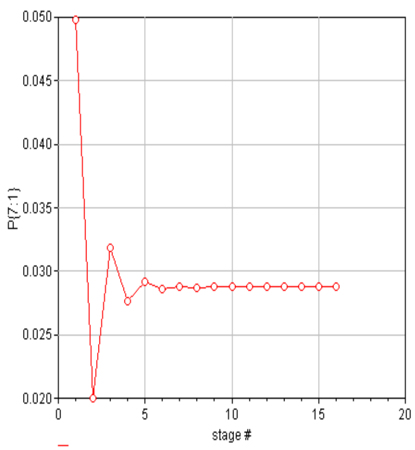
P^2 , the second power of the transition probability matrix, is

from \to	0	1	2	3	4	5	6
0	0.2696	0.1661	0.1848	0.1683	0.1237	0.06756	0.02004
1	0.2696	0.1661	0.1848	0.1683	0.1237	0.06756	0.02004
2	0.2696	0.1661	0.1848	0.1683	0.1237	0.06756	0.02004
3	0.1085	0.107	0.1671	0.2154	0.2129	0.1419	0.04731
4	0.1709	0.1254	0.168	0.1943	0.1819	0.1196	0.03987
5	0.2395	0.1502	0.175	0.1738	0.1441	0.08863	0.02872
6	0.2696	0.1661	0.1848	0.1683	0.1237	0.06756	0.02004

and the third power is

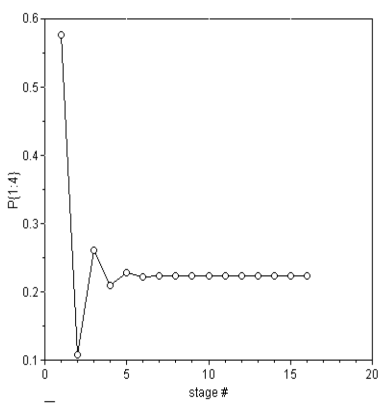
from \to	0	1	2	3	4	5	6
0	0.2069	0.1413	0.1756	0.1855	0.1597	0.09903	0.03189
1	0.2069	0.1413	0.1756	0.1855	0.1597	0.09903	0.03189
2	0.2069	0.1413	0.1756	0.1855	0.1597	0.09903	0.03189
3	0.2616	0.1631	0.1839	0.1706	0.1281	0.07126	0.0214
4	0.2406	0.1552	0.1813	0.1766	0.1399	0.08126	0.0251
5	0.2173	0.146	0.1778	0.183	0.1534	0.09305	0.02954
6	0.2069	0.1413	0.1756	0.1855	0.1597	0.09903	0.03189

Consider the behavior of $P_{6,0}^{(n)}$ as n increases:



n	$p_{6,0}^{(n)}$
1	0.0839179
2	0.269638
3	0.206912
4	0.228275
5	0.220989
6	0.223475
7	0.222627
8	0.222916
9	0.222818
10	0.222851
11	0.22284
12	0.222844
13	0.222842
14	0.222843
15	0.222843
16	0.222843

Compare to the behavior of $P_{4,0}^{(n)}$ as n increases:



n	$p_{4,0}^{(n)}$
1	0.57681
2	0.108458
3	0.261614
4	0.209636
5	0.227347
6	0.221306
7	0.223367
8	0.222664
9	0.222904
10	0.222822
11	0.22285
12	0.22284
13	0.222844
14	0.222842
15	0.222843
16	0.222843

The limiting distribution of the state of the system (independent of the initial state) is

State i	Steadystate Probability π_i
0	0.222843
1	0.14779
2	0.178159
3	0.181227
4	0.150444
5	0.0907828
6	0.0287543

This implies that 22.2843% of the days a stockout occurs, 14.779% of the days the ending inventory is 1, etc.

Recursive Computation of First-Passage Probabilities

$$f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k<n} p_{jj}^{(n-k)} f_{ij}^{(k)} \text{ where } f_{ij}^{(1)} \equiv p_{ij}$$

$$f_{6,0}^{(1)} = p_{6,0} = 0.08392$$

$$f_{6,0}^{(2)} = p_{6,0}^{(2)} - \left[p_{0,0}^{(1)} f_{6,0}^{(1)} \right] = 0.2696 - 0.08392 \times 0.08392 = 0.26256$$

$$f_{6,0}^{(3)} = p_{6,0}^{(3)} - \left[p_{0,0}^{(1)} f_{6,0}^{(2)} + p_{0,0}^{(2)} f_{6,0}^{(1)} \right] = 0.2069 - [0.08392 \times 0.26256 + 0.2696 \times 0.08392]$$

$$= 0.1622$$

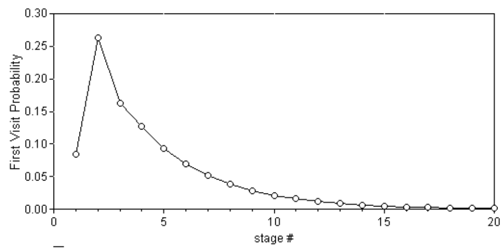
etc.

Thus, if the inventory is full on Monday morning (or equivalently, Sunday evening),

the probability that the ***first stockout*** occurs Wednesday evening (n=3) is 16.22%.

The probability distribution $f_{6,0}^{(n)}$ of the first-passage times are

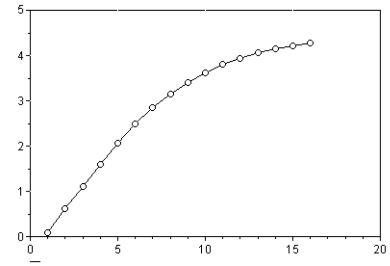
n	$f_{6,0}^{(n)}$
1	0.0839179
2	0.262596
3	0.162248
4	0.12649
5	0.0931353
6	0.0694925
7	0.0516882
8	0.0384743
9	0.0286333
10	0.0213104
11	0.0158602
12	0.0118039
13	0.00878497
14	0.00653818
15	0.00486601
16	0.00362151



Expected First-Passage Time

n	$f_{6,0}^{(n)}$	$n\times f_{6,0}^{(n)}$	$\sum_{k\le n} k\times f_{6,0}^{(k)}$
1	0.0839179	0.0839179	0.0839179
2	0.262596	0.525192	0.60911
3	0.162248	0.486745	1.09586
4	0.12649	0.50596	1.60182
5	0.0931353	0.465677	2.06749
6	0.0694925	0.416955	2.48445
7	0.0516882	0.361818	2.84627
8	0.0384743	0.307794	3.15406
9	0.0286333	0.2577	3.41176
10	0.0213104	0.213104	3.62486
11	0.0158602	0.174462	3.79933
12	0.0118039	0.141646	3.94097
13	0.00878497	0.114205	4.05518
14	0.00653818	0.0915345	4.14671
15	0.00486601	0.0729902	4.2197
16	0.00362151	0.0579441	4.27765

The partial sums in the last column are successive approximations of the ***mean first passage time*** $m_{6,0}$, but convergence is somewhat slow:



Generally, it is more computationally efficient to compute the mean first passage times by solving a (7×7) system of linear equations for $m_{i,0}$, $i = 0,1,...6$:

$$m_{ij} = 1 + \sum_{k\neq j} p_{ik} m_{kj},\ \forall\ i \in \mathrm{I}\ \&\ \text{fixed } j$$

$$\text{Fix j=0: } \begin{cases} m_{0,0} = 1 + p_{0,1}m_{1,0} + p_{0,2}m_{2,0} + p_{0,3}m_{3,0} + \cdots + p_{0,6}m_{6,0} \\ m_{1,0} = 1 + p_{1,1}m_{1,0} + p_{1,2}m_{2,0} + p_{1,3}m_{3,1} + \cdots + p_{1,6}m_{6,1} \\ etc. \\ m_{6,0} = 1 + p_{6,1}m_{1,0} + p_{6,2}m_{2,0} + p_{6,3}m_{3,1} + \cdots + p_{6,6}m_{6,1} \end{cases}$$

That is, we solve the 7×7 linear system for $m_{0,0}, m_{1,0}, \cdots, m_{6,0}$:

1	-0.100819	-0.168031	-0.224042	-0.224042	-0.149361	-0.0497871	1
0	0.899181	-0.168031	-0.224042	-0.224042	-0.149361	-0.0497871	1
0	-0.100819	0.831969	-0.224042	-0.224042	-0.149361	-0.0497871	1
0	-0.224042	-0.149361	0.950213	0	0	0	1
0	-0.224042	-0.224042	-0.149361	0.950213	0	0	1
0	-0.168031	-0.224042	-0.224042	-0.149361	0.950213	0	1
0	-0.100819	-0.168031	-0.224042	-0.224042	-0.149361	0.950213	1

The complete *mean first passage* matrix is found by solving 7 such linear systems (one per column):

from \to	0	1	2	3	4	5	6
0	4.48747	6.76636	5.61296	4.75467	5.77744	10.1005	34.7774
1	4.48747	6.76636	5.61296	4.75467	5.77744	10.1005	34.7774
2	4.48747	6.76636	5.61296	4.75467	5.77744	10.1005	34.7774
3	2.81583	6.22338	5.78307	5.51794	6.82984	11.1529	35.8298
4	3.61112	6.13803	5.36867	5.05969	6.64699	11.3183	35.9952
5	4.13554	6.39543	5.34364	4.73395	6.16983	11.0153	36.2693
6	4.48747	6.76636	5.61296	4.75467	5.77744	10.1005	34.7774

Thus, starting with a full inventory (state 6), the expected number of days until a stockout occurs (state 0) is 4.48747.

The ***mean recurrence times*** m_{ii} on the diagonal are the reciprocals of the steadystate probabilities, i.e., $m_{ii} = 1/\pi_i$.

For example, one should expect a stockout once every

$$m_{00} = 1/\pi_0 = \frac{1}{0.222843} = 4.48747 \text{ (days)}$$

and to find the inventory full (S=6) at the end of the day once every

$$m_{6,6} = 1/\pi_6 = \frac{1}{0.0287543} = 34.7774 \text{ (days)}$$