Stochastic Dynamic Programming

Inventory Replenishment with Uncertain Demand and Backorders

At the end of each day, the inventory position for a single product is reviewed, and production is scheduled for up to 5 units. It is assumed that production is completed in time to satisfy any demand the following day. Up to 3 units may be backordered, and maximum inventory level is 6.

Probability distribution P of demand

i	d[i]	name	P{d[i]}
1	0	Demand 0	0.15
2	1	Demand 1	0.2
3	2	Demand 2	0.3
4	3	Demand 3	0.2
5	4	Demand 4	0.15

State of system s = end-of-day stock position

i	s[i]	name
1	‾3	Back3
2	‾2	Back2
3	‾1	Back1
4	0	Empty
5	1	Stock1
6	2	Stock2
7	3	Stock3
8	4	Stock4
9	5	Stock5
10	6	Stock6

Decision x = production level

i	x[i]	name
1	0	Idle
2	1	Prod 1
3	2	Prod 2
4	3	Prod 3
5	4	Prod 4
6	5	Prod 5

Shortage cost = $10s^-$

$10 per unit short

Inventory holding cost $H(s) = 3s^+$

$3 per unit in end-of-day storage

Production cost $G(x)$

Setup cost = $6, marginal cost = $4/unit

Salvage value $3 per unit at end of planning horizon

Define

stage n = # of days remaining in planning period, $n = 1,2,\ldots N$

state $s = s^+ - s^-$ = inventory position, where

$s^+ = \max\{0, s\}$ is *stock on hand*,

$s^- = \max\{0, -s\}$ is number of *backorders*,

i.e., $s^+ + s^- = |s|$

decision x = production quantity

Optimal value $f_n(s)$=minimum expected cost of the final n days of the planning period

$$f_n(s) = \min_x \left\{ 3s^+ + 10s^- + G(x) + \sum_d p_d f_{n-1}(s + x - d) \right\}, \text{ n=1,2,...N}$$

$$f_0(s) = 3s^+ + 10s^- + G(s^-)$$

Recursive APL function

```
    ∇ z←F N;t;Current_cost;Next_state;Stock_cost;Prod_cost
      ⍝
      ⍝ Optimal Value Function for Stochastic Inventory Replenishment
      ⍝         with backordering
      ⍝
      :if N=0   ⍝ Terminal conditions
        ⍝  Must produce to fill any remaining backorders
        Current_cost←(G+(⍴G) Extend SHORT)[1+0⌈-s]+(-SALVAGE)[1+0⌈s]
        ⍝  Big penalty is appended to prevent infeasible states
        z←Current_cost,BIG
      :else
        Stock_cost←H[1+0⌈s]+SHORT[1+0⌈-s]
        Prod_cost←G[1+x]
        Current_cost←(Stock_cost∘.+Prod_cost)
        ⍝  Big penalty is added to force filling backorders
        Current_cost←Current_cost+BIG×0>s∘.+x
        ⍝  Next state of system
        Next_state←(⌊/s)⌈(⌈/s)⌊s∘.+x∘.-d
        ⍝  recursion
        z←P Minimize_E (Current_cost∘.+0×d)+(F N-1)[TRANSITION Next_state]
      :endif
    ∇
```

H is the vector (0, 3, 6, 9, 12, 15)
G is the vector (0, 10, 14, 18, 22, 26)
SHORT is the vector (0, 10, 20, 30)
SALVAGE is the vector (0, 3, 6, 9, 12, 15)

Recursive computations with time horizon N = 9 days

Recursion type: backward

$$f_1(s) = \min_x \left\{ 3s^+ + 10s^- + G(x) + \sum_d p_d f_0(s+x-d) \right\}$$

where $f_0(s) = 3s^+ + 10s^- + G(s^-)$

---**Stage 1**---

s \ x:	0	1	2	3	4	5	$f_1(s)$
¯3	9999.99	9999.99	9999.99	79.00	71.55	63.60	63.60
¯2	9999.99	9999.99	65.00	57.55	49.60	45.55	45.55
¯1	9999.99	51.00	43.55	35.60	31.55	30.00	30.00
0	31.00	29.55	21.60	17.55	16.00	17.00	16.00
1	22.55	20.60	16.55	15.00	16.00	17.00	15.00
2	13.60	15.55	14.00	15.00	16.00	17.45	13.60
3	8.55	13.00	14.00	15.00	16.45	18.50	8.55
4	6.00	13.00	14.00	15.45	17.50	20.45	6.00
5	6.00	13.00	14.45	16.50	19.45	23.00	6.00
6	6.00	13.45	15.50	18.45	22.00	26.00	6.00

*Note that **9999.99** is used to indicate an infeasible combination of state & decision (since any backorders must be filled immediately by production).*

Example calculations:

If inventory position is s= **–2**, i.e., there are 2 units backordered,

and x=3, i.e., we order up to level 1 by producing **3** units:

Holding cost at end of previous day: **$0**
Shortage cost at end of previous day: **$20**
Production cost during current day: **$18** (i.e., $6 setup + 3×$4/unit)
Expected costs at end of planning horizon:

Demand d	Probability P{d}	Final state s_0	Cost	P{d}×cost
0	0.15	1	–3	–0.45
1	0.2	0	0	0.00
2	0.3	–1	20	6.00
3	0.2	–2	34	6.80
4	0.15	–3	48	7.20
			TOTAL	$19.55

Total expected cost = $0 + $20 + $18 + $19.55 = $57.55

Note that if the final state is a backorder position, the cost includes not only the shortage cost but the cost of production to fill these backorders. If there is positive final inventory, a salvage value is received.

Given the values of $f_1(s)$ which have just been computed, we next compute the values of $f_2(s)$:

---**Stage 2**---

s \ x:	0	1	2	3	4	5	Minimum
¯3	9999.99	9999.99	9999.99	92.32	85.10	78.67	78.67
¯2	9999.99	9999.99	78.32	71.10	64.67	62.20	62.20
¯1	9999.99	64.33	57.10	50.67	48.20	48.09	48.09
0	44.33	43.10	36.67	34.20	34.09	35.64	34.09
1	36.10	35.67	33.20	33.09	34.64	36.65	33.09
2	28.67	32.20	32.09	33.64	35.65	38.38	28.67
3	25.20	31.09	32.64	34.65	37.38	41.00	25.20
4	24.09	31.64	33.65	36.38	40.00	44.00	24.09
5	24.64	32.65	35.38	39.00	43.00	47.00	24.64
6	25.65	34.38	38.00	42.00	46.00	50.00	25.65

Example calculations:

If inventory position is s = 1, i.e., there is one unit in stock,

and production quantity is x =2, i.e., we order up to level 3 by producing 2 units:

Holding cost at end of previous day: **$3**
Shortage cost at end of previous day: **$0**
Production cost during current day: **$14** ($6 setup + 2×$4/unit)

Expected costs at end of planning horizon:

Demand d	Probability P{d}	Resulting state s_1	Cost $f_1(s_1)$	P{d}×cost
0	0.15	3	8.55	1.28
1	0.2	2	13.60	2.75
2	0.3	1	15.00	4.50
3	0.2	0	16.00	3.20
4	0.15	−1	30.00	4.50
			TOTAL	**$6.20**

Total expected cost of the final 2 days = $3 + $0 + $14 + $16.20 = **$33.20**

Given $f_2(s)$, we next calculate $f_3(s)$:

---**Stage 3**---

s \ x:	0	1	2	3	4	5	Minimum
−3	9999.99	9999.99	9999.99	108.93	102.45	96.09	96.09
−2	9999.99	9999.99	94.93	88.45	82.09	79.47	79.47
−1	9999.99	80.93	74.45	68.09	65.47	64.99	64.99
0	60.93	60.45	54.09	51.47	50.99	52.77	50.99
1	53.45	53.09	50.47	49.99	51.77	54.34	49.99
2	46.09	49.47	48.99	50.77	53.34	56.97	46.09
3	42.47	47.99	49.77	52.34	55.97	60.21	42.47
4	40.99	48.77	51.34	54.97	59.21	63.50	40.99
5	41.77	50.34	53.97	58.21	62.50	66.65	41.77
6	43.34	52.97	57.21	61.50	65.65	69.65	43.34

---**Stage 4**---

s \ x:	0	1	2	3	4	5	Minimum
−3	9999.99	9999.99	9999.99	126.12	119.50	113.13	113.13
−2	9999.99	9999.99	112.12	105.50	99.13	96.53	96.53
−1	9999.99	98.12	91.50	85.13	82.53	82.12	82.12
0	78.12	77.50	71.13	68.53	68.12	69.92	68.12
1	70.50	70.13	67.53	67.12	68.92	71.56	67.12
2	63.13	66.53	66.12	67.92	70.56	74.27	63.13
3	59.53	65.12	66.92	69.56	73.27	77.68	59.53
4	58.12	65.92	68.56	72.27	76.68	81.11	58.12
5	58.92	67.56	71.27	75.68	80.11	84.34	58.92
6	60.56	70.27	74.68	79.11	83.34	87.34	60.56

---**Stage 5**---

s \ x:	0	1	2	3	4	5	Minimum
−3	9999.99	9999.99	9999.99	143.19	136.60	130.23	130.23
−2	9999.99	9999.99	129.19	122.60	116.23	113.63	113.63
−1	9999.99	115.19	108.60	102.23	99.63	99.20	99.20
0	95.19	94.60	88.23	85.63	85.20	87.01	85.20
1	87.60	87.23	84.63	84.20	86.01	88.68	84.20
2	80.23	83.63	83.20	85.01	87.68	91.43	80.23
3	76.63	82.20	84.01	86.68	90.43	94.87	76.63
4	75.20	83.01	85.68	89.43	93.87	98.32	75.20
5	76.01	84.68	88.43	92.87	97.32	101.56	76.01
6	77.68	87.43	91.87	96.32	100.56	104.56	77.68

---Stage 6---

s \ x:	0	1	2	3	4	5	Minimum
¯3	9999.99	9999.99	9999.99	160.29	153.69	147.32	147.32
¯2	9999.99	9999.99	146.29	139.69	133.32	130.72	130.72
¯1	9999.99	132.29	125.69	119.32	116.72	116.30	116.30
0	112.29	111.69	105.32	102.72	102.30	104.11	102.30
1	104.69	104.32	101.72	101.30	103.11	105.78	101.30
2	97.32	100.72	100.30	102.11	104.78	108.53	97.32
3	93.72	99.30	101.11	103.78	107.53	111.97	93.72
4	92.30	100.11	102.78	106.53	110.97	115.43	92.30
5	93.11	101.78	105.53	109.97	114.43	118.68	93.11
6	94.78	104.53	108.97	113.43	117.68	121.68	94.78

---Stage 7--

s \ x:	0	1	2	3	4	5	Minimum
¯3	9999.99	9999.99	9999.99	177.38	170.79	164.41	164.41
¯2	9999.99	9999.99	163.38	156.79	150.41	147.82	147.82
¯1	9999.99	149.38	142.79	136.41	133.82	133.39	133.39
0	129.38	128.79	122.41	119.82	119.39	121.20	119.39
1	121.79	121.41	118.82	118.39	120.20	122.87	118.39
2	114.41	117.82	117.39	119.20	121.87	125.62	114.41
3	110.82	116.39	118.20	120.87	124.62	129.07	110.82
4	109.39	117.20	119.87	123.62	128.07	132.53	109.39
5	110.20	118.87	122.62	127.07	131.53	135.78	110.20
6	111.87	121.62	126.07	130.53	134.78	138.78	111.87

---Stage 8---

s \ x:	0	1	2	3	4	5	Minimum
¯3	9999.99	9999.99	9999.99	194.48	187.88	181.51	181.51
¯2	9999.99	9999.99	180.48	173.88	167.51	164.91	164.91
¯1	9999.99	166.48	159.88	153.51	150.91	150.48	150.48
0	146.48	145.88	139.51	136.91	136.48	138.29	136.48
1	138.88	138.51	135.91	135.48	137.29	139.96	135.48
2	131.51	134.91	134.48	136.29	138.96	142.71	131.51
3	127.91	133.48	135.29	137.96	141.71	146.16	127.91
4	126.48	134.29	136.96	140.71	145.16	149.62	126.48
5	127.29	135.96	139.71	144.16	148.62	152.87	127.29
6	128.96	138.71	143.16	147.62	151.87	155.87	128.96

---Stage 9---

s \ x:	0	1	2	3	4	5	Minimum
¯3	9999.99	9999.99	9999.99	211.57	204.97	198.60	198.60
¯2	9999.99	9999.99	197.57	190.97	184.60	182.00	182.00
¯1	9999.99	183.57	176.97	170.60	168.00	167.57	167.57
0	163.57	162.97	156.60	154.00	153.57	155.39	153.57
1	155.97	155.60	153.00	152.57	154.39	157.06	152.57
2	148.60	152.00	151.57	153.39	156.06	159.81	148.60
3	145.00	150.57	152.39	155.06	158.81	163.26	145.00
4	143.57	151.39	154.06	157.81	162.26	166.71	143.57
5	144.39	153.06	156.81	161.26	165.71	169.96	144.39
6	146.06	155.81	160.26	164.71	168.96	172.96	146.06

Typically, the state at the beginning of the process is known, so that only one row of this table need be computed.

For example, if we begin with zero inventory, the minimum total expected cost for the nine-day planning horizon is $153.57

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Stage 9

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	198.60
Back2	Prod 5	182.00
Back1	Prod 5	167.57
Empty	Prod 4	153.57
Stock1	Prod 3	152.57
Stock2	Idle	148.60
Stock3	Idle	145.00
Stock4	Idle	143.57
Stock5	Idle	144.39
Stock6	Idle	146.06

Optimal Policy

The optimal policy is: if the stock is 1 or less, produce enough to increase the stock level to 4 (up to a maximum production quantity of 5). *Note that this is approximately an (s,S) = (1,4) replenishment policy.*

The decisions at stages 8, 7, ….1 cannot be determined until the states at those stages have been observed!

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Stage 8

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	181.51
Back2	Prod 5	164.91
Back1	Prod 5	150.48
Empty	Prod 4	136.48
Stock1	Prod 3	135.48
Stock2	Idle	131.51
Stock3	Idle	127.91
Stock4	Idle	126.48
Stock5	Idle	127.29
Stock6	Idle	128.96

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Stage 7

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	164.41
Back2	Prod 5	147.82
Back1	Prod 5	133.39
Empty	Prod 4	119.39
Stock1	Prod 3	118.39
Stock2	Idle	114.41
Stock3	Idle	110.82
Stock4	Idle	109.39
Stock5	Idle	110.20
Stock6	Idle	111.87

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The optimal policies for stages 8 and 7 are identical with that of stage 9!

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Stage 6

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	147.32
Back2	Prod 5	130.72
Back1	Prod 5	116.30
Empty	Prod 4	102.30
Stock1	Prod 3	101.30
Stock2	Idle	97.32
Stock3	Idle	93.72
Stock4	Idle	92.30
Stock5	Idle	93.11
Stock6	Idle	94.78

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Stage 5

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	130.23
Back2	Prod 5	113.63
Back1	Prod 5	99.20
Empty	Prod 4	85.20
Stock1	Prod 3	84.20
Stock2	Idle	80.23
Stock3	Idle	76.63
Stock4	Idle	75.20
Stock5	Idle	76.01
Stock6	Idle	77.68

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Stage 4

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	113.13
Back2	Prod 5	96.53
Back1	Prod 5	82.12
Empty	Prod 4	68.12
Stock1	Prod 3	67.12
Stock2	Idle	63.13
Stock3	Idle	59.53
Stock4	Idle	58.12
Stock5	Idle	58.92
Stock6	Idle	60.56

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Stage 3

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	96.09
Back2	Prod 5	79.47
Back1	Prod 5	64.99
Empty	Prod 4	50.99
Stock1	Prod 3	49.99
Stock2	Idle	46.09
Stock3	Idle	42.47
Stock4	Idle	40.99
Stock5	Idle	41.77
Stock6	Idle	43.34

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Stage 2

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	78.67
Back2	Prod 5	62.20
Back1	Prod 5	48.09
Empty	Prod 4	34.09
Stock1	Prod 3	33.09
Stock2	Idle	28.67
Stock3	Idle	25.20
Stock4	Idle	24.09
Stock5	Idle	24.64
Stock6	Idle	25.65

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Stage 1

Current State	Optimal Decision	Optimal Value
Back3	Prod 5	63.60
Back2	Prod 5	45.55
Back1	Prod 5	30.00
Empty	Prod 4	16.00
Stock1	Prod 3	15.00
Stock2	Idle	13.60
Stock3	Idle	8.55
Stock4	Idle	6.00
Stock5	Idle	6.00
Stock6	Idle	6.00

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Optimal Policy

Only in the final day is the optimal policy different!