CYCLIC STAFFING

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Cyclic Staffing problems are characterized by

- n = # periods per cycle
- m = # periods per shift
- C_i = cost per worker for shift i
- R_j = number of workers req'd in period j

For example, for n=7, m=5, each worker's shift consists of 5 consecutive periods (days) per cycle (week).

Staffing requirements, as well as the cost per worker, are given for each period during the cycle.

The problem is to determine the number of workers to be assigned to each shift.

Example The Kleen City Police Department is preparing a shift schedule for the policemen & policewomen.

The 24-hour day is divided into six 4-hour periods, with the first period beginning at 2:00 am.

Each person works two consecutive 4-hour periods, i.e., 8 consecutive hours.

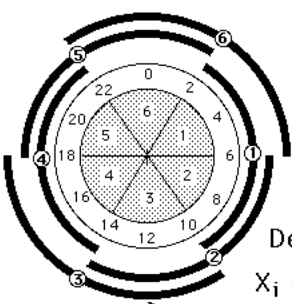
n=6
m=2

The requirements in each period (which are the same for each day of the week) are: R_j

period #	time of day	requirement
1	02-06	22
2	06-10	55
3	10-14	88
4	14-18	110
5	18-22	44
6	22-02	33

times are according to 24-hour clock!

We wish a daily plan which employs the least number of persons. $C_i = 1$

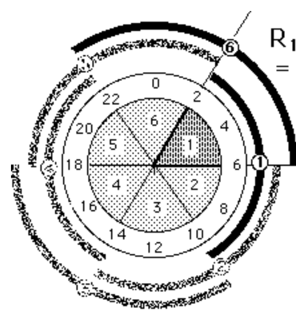


indicates shift

There are six possible shifts that a person may work. Let shift #i be the shift starting in period #i and including period #i+1.

Define **decision variables**

X_i = # of persons assigned to shift #i (i.e., working in periods i & i+1)



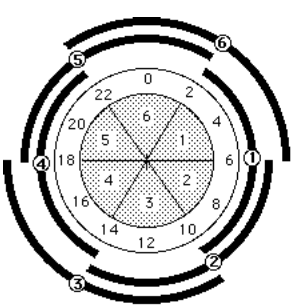
R_1 =22

Constraints

# of person working in period #i (i.e., shifts #i and #i-1) must be at least the number required.

For example, persons who are assigned to shifts 1&6 work in period #1, and so

$$X_1 + X_6 \geq 22 = R_1$$



Minimize

$$X_1+X_2+X_3+X_4+X_5+X_6$$

subject to

$$\begin{aligned}
X_1 \quad\quad\quad\quad + X_6 &\geq 22 = R_1\\
X_1+X_2 \quad\quad\quad &\geq 55 = R_2\\
X_2+X_3 \quad\quad &\geq 88 = R_3\\
X_3+X_4 \quad &\geq 110 = R_4\\
X_4+X_5 &\geq 44 = R_5\\
X_5+X_6 &\geq 33 = R_6
\end{aligned}$$

$$X_i \geq 0 \text{ \& integer}$$

Minimize

$$X_1+X_2+X_3+X_4+X_5+X_6$$

subject to

staffing requirements

nonnegativity constraints

$$\begin{bmatrix}
1&0&0&0&0&1\\
1&1&0&0&0&0\\
0&1&1&0&0&0\\
0&0&1&1&0&0\\
0&0&0&1&1&0\\
0&0&0&0&1&1\\
\hline
1&0&0&0&0&0\\
0&1&0&0&0&0\\
0&0&1&0&0&0\\
0&0&0&1&0&0\\
0&0&0&0&1&0\\
0&0&0&0&0&1
\end{bmatrix} X \geq \begin{bmatrix}
22\\55\\88\\110\\44\\33\\ \hline 0\\0\\0\\0\\0\\0
\end{bmatrix}$$

Change of variable

$$\begin{array}{l} Y_1 = X_1 \\ Y_2 = X_1+X_2 \\ Y_3 = X_1+X_2+X_3 \\ Y_4 = X_1+X_2+X_3+X_4 \\ Y_5 = X_1+X_2+X_3+X_4+X_5 \\ Y_6 = X_1+X_2+X_3+X_4+X_5+X_6 \end{array} \quad \Leftrightarrow \quad \begin{array}{l} X_1 = Y_1 \\ X_2 = Y_2 - Y_1 \\ X_3 = Y_3 - Y_2 \\ X_4 = Y_4 - Y_3 \\ X_5 = Y_5 - Y_4 \\ X_6 = Y_6 - Y_5 \end{array}$$

Minimize Y_6

subject to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} Y \geq \begin{bmatrix} 22 \\ 55 \\ 88 \\ 110 \\ 44 \\ 33 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

staffing requirements (top six rows)

nonnegativity constraints (bottom six rows)

(Y unrestricted in sign, but integer!)

There appears to be some similarity to a node-arc incidence matrix of a network, namely the elements consist only of +1, -1, and zero!

The transpose of the matrix would appear even more similar.... many rows have only two nonzero elements (+1 & -1)!

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

In fact, if not for the "1" in the upper-right corner, the matrix transpose would be a node-arc incidence matrix!

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & \boxed{1} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Minimize Y_6 subject to

The problem can be viewed as finding the minimum Y_6 such that the constraints are feasible:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} \geq \begin{bmatrix} 22 \\ 55 \\ 88 \\ 110 \\ 44 \\ 33 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} Y_6$$

(Y unrestricted in sign, but integer!)

Suppose that Y_6 is temporarily fixed... the dual LP then becomes

$$\text{Max } (22-Y_6)\pi_1 + 55\pi_2 + 88\pi_3 + 110\pi_4 + 44\pi_5 + (33Y_6)\pi_6 - Y_6\mu_6$$

subject to

$$\left[\begin{array}{cccccc|cccccc} 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{bmatrix} \pi \\ \hdashline \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\pi_i \geq 0, i=1,2,...6$ $\mu_i \geq 0, i=1,2,...6$

The coefficient matrix is a node-arc incidence matrix if a redundant constraint is added, namely the negative of the sum of the five equations:

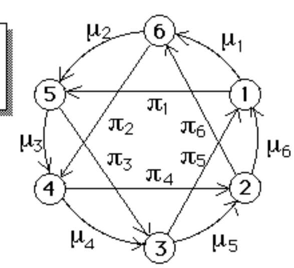
$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \left[\begin{array}{cccccc|cccccc} 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \pi \\ \hdashline \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

NEW ROW (row 6)

For fixed values of Y_6, we would need to solve the network problem:

$$\text{Max } (22-Y_6)\pi_1 + 55\pi_2 + 88\pi_3 + 110\pi_4 + 44\pi_5 + (33Y_6)\pi_6 - Y_6\mu_6$$

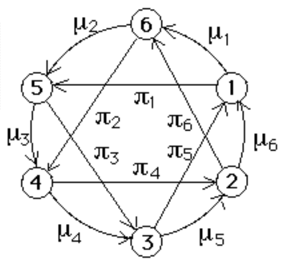
supply/demand at each of the six nodes is zero!



If there exists any feasible flow having a positive objective value, the LP is *unbounded* (which implies that the primal LP is *infeasible* !)

$$\text{Max } (22-Y_6)\pi_1 + 55\pi_2 + 88\pi_3 + 110\pi_4 + 44\pi_5 + (33Y_6)\pi_6 - Y_6\mu_6$$

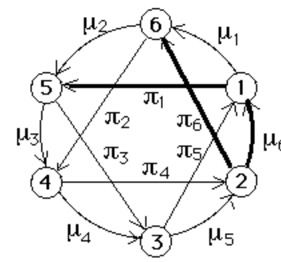
supply/demand at each of the six nodes is zero!



$$\text{Max } (22-Y_6)\pi_1 + 55\pi_2 + 88\pi_3 + 110\pi_4 + 44\pi_5 + (33Y_6)\pi_6 - Y_6\mu_6$$

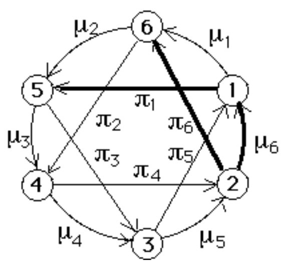
The LP is unbounded if there exists a cycle with total "length" which is positive!

"bold" arcs have length which depends upon Y_6



Minimize Y_6 + $\text{Max } (22-Y_6)\pi_1 + 55\pi_2 + 88\pi_3 + 110\pi_4 + 44\pi_5 + (33Y_6)\pi_6 - Y_6\mu_6$

The problem becomes that of finding the smallest value of such that there exists no positive-length cycle in the network!



A 0/1 matrix has

properly compatible circular 1's in columns

if

- the 1's in each column are circular
- if the first "1" (in a cyclic sense) of column j precedes that of column k, then the last "1" (in a cyclic sense) of column k does not precede that of column j.

Both of the matrices below have "circular 1's in columns", but are not "properly compatible":

$$\begin{bmatrix} \cdots & 0\ 0 & \cdots \\ \cdots & 1\ 0 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 0 & \cdots \\ \cdots & 0\ 0 & \cdots \\ \cdots & 0\ 0 & \cdots \end{bmatrix} \qquad \begin{bmatrix} \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 0 & \cdots \\ \cdots & 0\ 0 & \cdots \\ \cdots & 0\ 0 & \cdots \\ \cdots & 0\ 0 & \cdots \\ \cdots & 1\ 0 & \cdots \\ \cdots & 1\ 1 & \cdots \end{bmatrix}$$

shift begins later & ends earlier than the previous shift

$$\begin{bmatrix} \cdots & 0\ 0 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 0\ 1 & \cdots \\ \cdots & 1\ 0 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 1\ 1 & \cdots \\ \cdots & 0\ 1 & \cdots \\ \cdots & 0\ 0 & \cdots \end{bmatrix}$$

lunch break *lunch break*

Suppose that each shift consists of 4 consecutive hours, with a 1-hour lunch break, plus another 4 consecutive hours....

Properly compatible, but not circular 1's in columns

Our example problem (Kleen City Police Dept.) does have "properly compatible circular 1's in columns":

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

If the matrix for a cyclic staffing problem has properly compatible circular 1's in columns, then the variable transformation

$X_1 = Y_1$
$X_2 = Y_2 - Y_1$
$X_3 = Y_3 - Y_2$
$X_4 = Y_4 - Y_3$
etc.

results in a problem whose dual, for fixed integer values of Y_6, is a network flow problem (with integer solution).

For the special case $C_i = 1$ for all $i=1,\ldots n$,

i.e., objective is to minimize the total number of workers,

the problem may be solved by making the transformation to $Y_1, Y_2, \ldots Y_n$, solving the continuous LP relaxation, and rounding each of the non-integer Y_i's up to the next integer!

Reference

Bartholdi, John J., III, Orlin, James B., and Ratliff, Donald, "Cyclic Scheduling via Integer Programs with Circular Ones",

Operations Research, Vol. 28, No. 5 (Sept.-Oct. 1980), pp. 1074-1085.