Convexity of Sets & Functions

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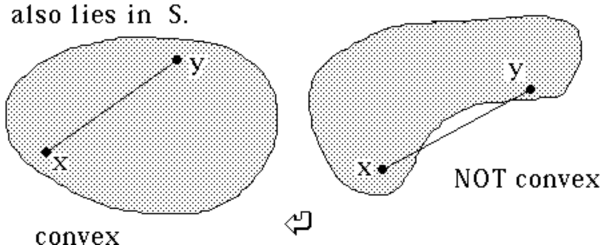
The property of "CONVEXITY" of sets and of functions is central to most approaches to nonlinear programming.

- Convex Set
- Convex Function

The more "unified" approach first defines convexity of sets, and bases the definition of convex function upon convexity of sets.

Convex Set

A set S is CONVEX if for every pair of elements x and y in S, the line segment joining x and y also lies in S.

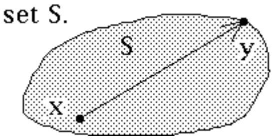


This means that, if we are solving the problem

Minimize $f(x)$
subject to $x \in S$

and S is convex, that there is a linear path such that we can follow this path directly from the starting point $x \in S$ to the optimal solution $y \in S$ without leaving the set S.

Unfortunately, we are seldom able to determine this line!



The *line segment* between x and y is given by

$$\lambda y + (1-\lambda)x = x + \lambda(y-x) \quad \text{for } \lambda \in [0,1]$$

where

$\lambda=0 \Rightarrow \lambda y + (1-\lambda)x = x$

$\lambda=1 \Rightarrow \lambda y + (1-\lambda)x = y$

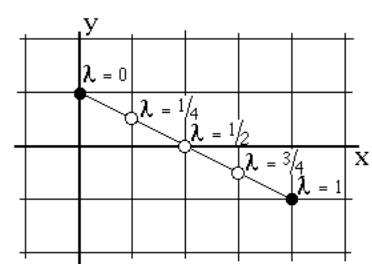
$\lambda=\frac{1}{2} \Rightarrow \lambda y + (1-\lambda)x = \frac{1}{2}(x+y)$ *(midpt of segment)*,

etc.

Example — *line segment*

Let $x=(0,1)$ and $y=(4,-1)$ be points in the plane.

λ	$\lambda y + (1-\lambda)x$
0	(0,1)
0.25	(1, 0.5)
0.50	(2, 0)
0.75	(3, -0.5)
1	(4, -1)



Convex Combination

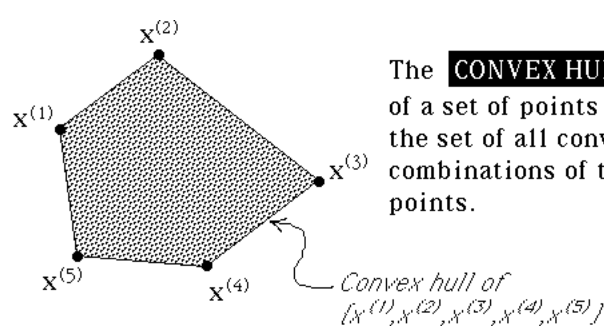
If $x^{(1)}, x^{(2)}, \ldots x^{(k)}$ are vectors in R^n, and $\lambda_1, \lambda_2, \cdots \lambda_k$ are nonnegative numbers whose sum is 1, i.e., $\sum_{i=1}^{k} \lambda_i = 1$

then $\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \ldots \lambda_k x^{(k)}$

is a convex combination (weighted average) of

$x^{(1)}, x^{(2)}, \ldots x^{(k)}$

In particular, a point on a line segment is a convex combination of the endpts. of the segment!



The CONVEX HULL of a set of points is the set of all convex combinations of those points.

Theorem If $x^{(1)}, x^{(2)}, \ldots x^{(k)} \in S$ where

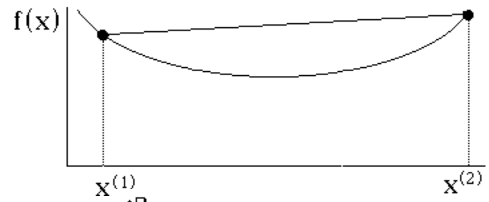
S is a convex set, then every convex combination of the points $x^{(1)}, x^{(2)}, \ldots x^{(k)}$ is an element of S.

That is, if S is convex then S equals its convex hull.

Convex Function A function f(x) is ***convex*** if:

$$f\left(\lambda x^{(1)} + (1-\lambda)x^{(2)}\right) \le \lambda f\left(x^{(1)}\right) + (1-\lambda)\, f\left(x^{(2)}\right) \quad \forall\, \lambda \in [0,1]$$

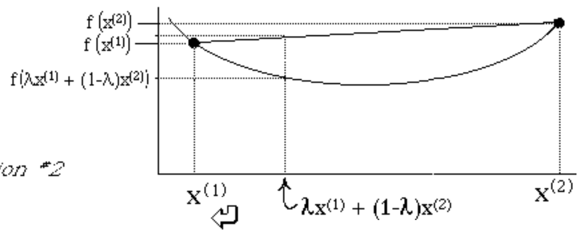
For example, f evaluated at the midpoint of two points is less than the average of the function values at the two points.



Definition #1

Convex Function A function f(x) is ***convex*** if:

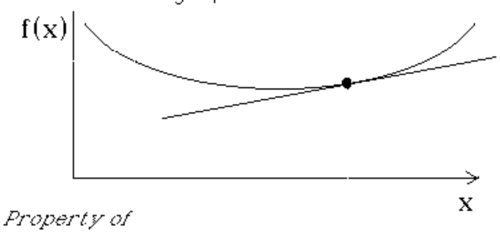
$$f\left(\lambda x^{(1)} + (1-\lambda)x^{(2)}\right) \le \lambda f\left(x^{(1)}\right) + (1-\lambda)\, f\left(x^{(2)}\right) \quad \forall\, \lambda \in [0,1]$$



Definition #2

Convex Function A differentiable function f(x) is ***convex*** if:

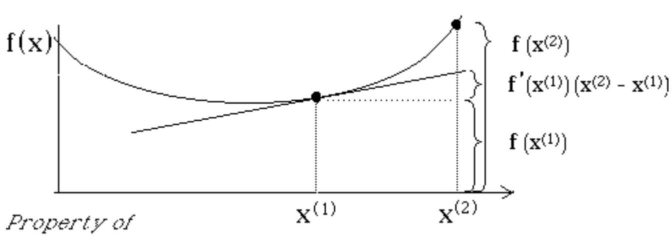
the tangent line (hyperplane) to the graph lies on or below the graph:



Property of convex function

Convex Function A differentiable function f(x) is ***convex*** if:

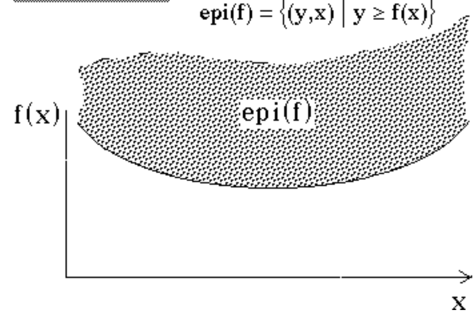
$$f\left(x^{(1)}\right) + f'\left(x^{(1)}\right)\left(x^{(2)} - x^{(1)}\right) \le f\left(x^{(2)}\right)$$



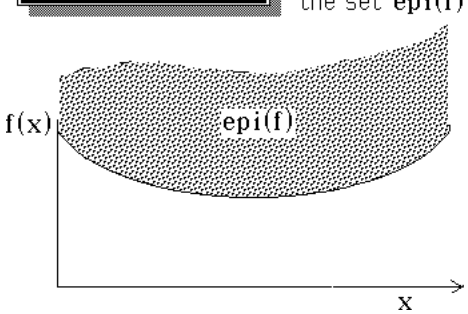
Property of convex function

Epigraph The epigraph of a function is the set

$$\mathbf{epi(f)} = \left\{(y,x) \mid y \ge f(x)\right\}$$



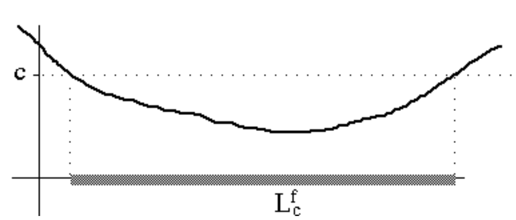
Convex Function A function f(x) is ***convex*** if the set **epi(f)** is convex



relationship between the concepts of convexity of a set and of a function!

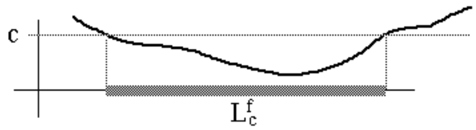
For any real value c, the **Level Set** of the function f is the set

$$L_c^f = \{x \mid f(x) \le c\}$$



If f is a convex function, then L_c^f is convex.

However, the convexity of the level sets does NOT imply convexity of the function.



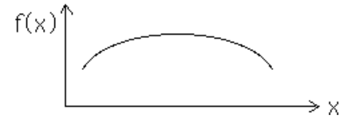
Example

If all the level sets of a function are convex, then the function is *quasi*convex.

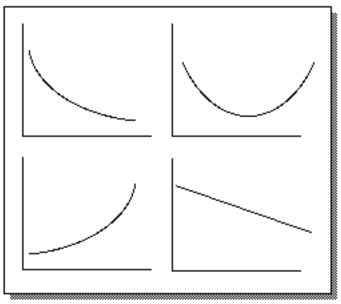
Concave Function

A function f is ***concave*** if

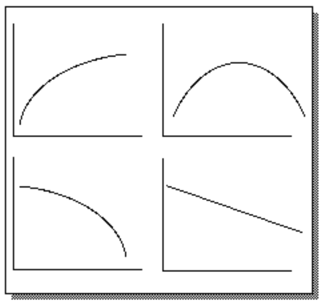
- its negative, (−f), is convex
- a chord between 2 points on the graph lies on or below the graph
- a tangent line (hyperplane) to the graph lies on or above the graph
- the hypergraph $\{(y,x) \mid y \le f(x)\}$ is convex



Examples



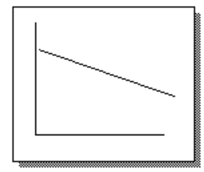
convex functions



concave functions

Examples

A linear function is ***both*** convex and concave!



A function may be neither convex nor concave: