Chance-Constrained LP

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A "chance constraint" is a modification of a constraint in which the right-hand-side is *random*.

Rather than guaranteeing that the constraint is satisfied for every possible right-hand-side value (which may be impossible, if the random variable is unbounded), a restriction is imposed that the constraint be satisfied by the optimal solution with ***at least*** a certain specified probability.

Consider the constraint

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$

where b_i is a random variable.

For example, suppose x_j *is the production time for process j, and* a_{ij} *is the consumption rate of raw material i by process j. The right-hand-side* b_i *could be the (random) quantity of resource i which will be available.*

The above constraint requires that the scheduled production time by the processes not consume more raw material than will be available.

If x_j must be selected ***before*** the value of b_i is known, then to guarantee satisfaction of the constraint, we would need to require that

$$\sum_{j=1}^{n} a_{ij} x_j \le \underline{b}_i$$

where $\underline{b}_i$ is the minimum possible value of b_i.

This may be overly restrictive, e.g., when b_i has a normal distribution, $\underline{b}_i = -\infty$ which may be impossible to satisfy, or in most cases, $\underline{b}_i = 0$, which might be satisfied only by $x = 0$

CHANCE CONSTRAINT

$$P\left\{\sum_{j=1}^{n} a_{ij} x_j \le b_i\right\} \ge \alpha$$

i.e., we require that the original constraint

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$

be satisfied with at least probability α.

As stated, this is not a valid LP constraint!

LINEARIZING A CHANCE CONSTRAINT

Given the distribution function (cdf)

$$F_i(y) = P\{b_i \le y\}$$

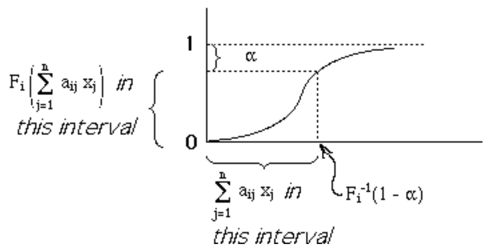
our chance constraint is equivalent to

$$P\left\{\sum_{j=1}^{n} a_{ij} x_j \le b_i\right\} = 1 - P\left\{b_i \le \sum_{j=1}^{n} a_{ij} x_j\right\} = 1 - F_i\left(\sum_{j=1}^{n} a_{ij} x_j\right)$$

i.e., $1 - F_i\left(\sum_{j=1}^{n} a_{ij} x_j\right) \ge \alpha$ **or** $F_i\left(\sum_{j=1}^{n} a_{ij} x_j\right) \le 1 - \alpha$

But $F_i\left(\sum_{j=1}^{n} a_{ij} x_j\right) \le 1 - \alpha \iff F_i^{-1}(1-\alpha) \ge \sum_{j=1}^{n} a_{ij} x_j$

The inequality on the right is linear!



EXAMPLE

Water Resources Planning Under Uncertainty

A water system manager must allocate water from a stream to three users:

- municipality
- industrial concern
- agricultural sector

Use	Request	Net Benefit per unit
1. Municipality	2	100
2. Industrial	3	50
3. Agricultural	5	30

Let X_i = amount of water allocated to use #i

The optimal allocation might be found by solving the LP:

$$\begin{aligned} \text{Max } & 100X_1 + 50X_2 + 30X_3 \\ \text{subject to } & X_1 + X_2 + X_3 \le Q \\ & 0 \le X_1 \le 2 \\ & 0 \le X_2 \le 3 \\ & 0 \le X_3 \le 5 \end{aligned}$$

But the decision must be made before the quantity Q of the available water is known!

$$\begin{aligned} \text{Max } & 100X_1 + 50X_2 + 30X_3 \\ \text{subject to } & X_1 + X_2 + X_3 \le Q \\ & 0 \le X_1 \le 2 \\ & 0 \le X_2 \le 3 \\ & 0 \le X_3 \le 5 \end{aligned}$$

Random variable with known probability distribution, namely, N(7,1.5) i.e., normal, with mean μ=7 and std deviation σ=1.5.

How should the water be allocated before the quantity available is known?

$$X_1 + X_2 + X_3 \le Q$$

$$P\left\{Q \ge X_1 + X_2 + X_3\right\} \ge \alpha$$

$$\Leftrightarrow \quad 1 - F(X_1 + X_2 + X_3) \ge \alpha$$

$$\Leftrightarrow \quad F(X_1 + X_2 + X_3) \le 1 - \alpha$$

$$\Leftrightarrow \quad X_1 + X_2 + X_3 \le F^{-1}(1 - \alpha)$$

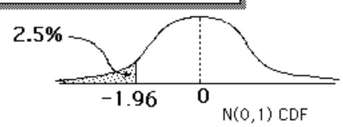
Suppose

α =97.5%

μ=7

σ=1.5

$$\Rightarrow \quad X_1 + X_2 + X_3 \le \mu - 1.96\,\sigma = 4.06$$



LINDO

```
MAX     150 X1 + 50 X2 + 30 X3
SUBJECT TO
       2)   X1 + X2 + X3 <=   4.06
END
SUB       X1         2.00000
SUB       X2         3.00000
SUB       X3         5.00000

    OBJECTIVE FUNCTION VALUE
    1)    403.000000

VARIABLE        VALUE     REDUCED COST
    X1         2.0000     -100.0000
    X2         2.0600         .0000
    X3          .0000       20.0000

   ROW   SLACK OR SURPLUS  DUAL PRICES
    2)            .0000       50.0000
```


JOINT CHANCE CONSTRAINTS

Suppose that the RHSs of several constraints are random:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for i=1, 2, ... k}$$

We might impose a chance constraint for ***each*** of the k random right-hand-sides

$$\sum_{j=1}^{n} a_{ij} x_j \le F_i^{-1}(1 - \alpha) \quad \text{for i=1, 2, ... k}$$

These chance constraints will ***not*** guarantee that the optimal solution is feasible with probability α.

Rather, if the right-hand-sides are independent random variables, then the optimal x would satisfy ***all*** of the constraints with probability α^k.

For example, if α = 95% and there are k=10 chance constraints, then x is feasible with probability α^k = 59.9%

Assume that the k random variables are independent, and that we require

$$P\left\{\left[\sum_{j=1}^{n} a_{1j} x_j \le b_1\right] \text{and} \left[\sum_{j=1}^{n} a_{2j} x_j \le b_2\right] \text{and} \cdots \left[\sum_{j=1}^{n} a_{kj} x_j \le b_k\right]\right\} \ge \alpha$$

$$\Updownarrow$$

$$P\left[\sum_{j=1}^{n} a_{1j} x_j \le b_1\right] \times P\left[\sum_{j=1}^{n} a_{2j} x_j \le b_2\right] \times \cdots \times P\left[\sum_{j=1}^{n} a_{kj} x_j \le b_k\right] \ge \alpha$$

$$\Updownarrow$$

$$\left[1 - F_1\left(\sum_{j=1}^{n} a_{1j} x_j\right)\right] \times \left[1 - F_2\left(\sum_{j=1}^{n} a_{2j} x_j\right)\right] \times \cdots \times \left[1 - F_k\left(\sum_{j=1}^{n} a_{kj} x_j\right)\right] \ge \alpha$$

For example, if b_i has an exponential distribution with mean $1/\lambda_i$, i.e.,

$$F_i(y) = 1 - e^{-\lambda_i y}$$

then the joint chance-constraint has the form

$$\left[\exp\left(-\lambda_1 \sum_{j=1}^{n} a_{1j} x_j\right)\right] \times \left[\exp\left(-\lambda_2 \sum_{j=1}^{n} a_{2j} x_j\right)\right] \times \cdots \times \left[\exp\left(-\lambda_k \sum_{j=1}^{n} a_{kj} x_j\right)\right] \ge \alpha$$

which is a highly ***nonlinear*** constraint.

$$\left[\exp\left(-\lambda_1\sum_{j=1}^{n}a_{1j}x_j\right)\right]\times\left[\exp\left(-\lambda_2\sum_{j=1}^{n}a_{2j}x_j\right)\right]\times\cdots\times\left[\exp\left(-\lambda_k\sum_{j=1}^{n}a_{kj}x_j\right)\right]\geq\alpha$$

By using a log transformation, we can simplify to

$$\ln\left[\exp\left(-\lambda_1\sum_{j=1}^{n}a_{1j}x_j\right)\right]+\cdots+\ln\left[\exp\left(-\lambda_k\sum_{j=1}^{n}a_{kj}x_j\right)\right]\geq\ln\ \alpha$$

or

$$\left(-\lambda_1\sum_{j=1}^{n}a_{1j}x_j\right)+\left(-\lambda_2\sum_{j=1}^{n}a_{2j}x_j\right)+\cdots+\left(-\lambda_k\sum_{j=1}^{n}a_{kj}x_j\right)\geq\ln\ \alpha$$

$$\Rightarrow \boxed{\sum_{j=1}^{n}\sum_{i=1}^{k}\left(-a_{ij}\lambda_i\right)x_j\geq\ln\alpha}$$

which is, in fact linear!

In cases other than the exponential distribution, however, the constraint ***cannot*** be linearized by a log transformation.

In the case of the normal distribution, the constraint will remain nonlinear, and cannot even be written in closed form!

Frequently, however, the nonlinear constraint will have a ***convex*** feasible region, e.g. when $\mathbf{b}_i$'s have normal, gamma, or uniform distributions, so that multiple local optima don't exist.