Center Problems in a Network

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author

Center of a Network

Define the function $\sigma(x) = \underset{j\in N}{\text{maximum}}\ d(x,j)$

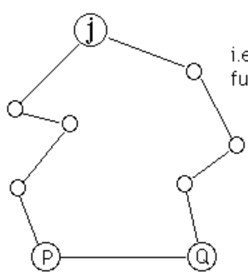
where

$d(x,j)$ = shortest path from x to node j

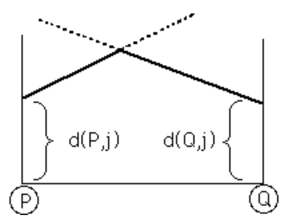
i.e., the distance from x to the farthest node of the network.

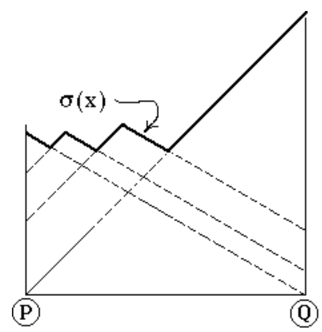
Suppose x ∈ edge [P,Q]

$d(x,j)$ = shortest path from x to node j
= minimum of d(X,P)+d(P,j)
and d(X,Q)+d(Q,j)



i.e., the lower envelope of two linear functions





$= \underset{j\in N}{\text{maximum}}\ d(x,j)$

For x ∈ edge [P,Q],

$\sigma(x)$ is the upper envelope of the functions d(X,j) for j ∈ N

The *Vertex Center* is the point x∈N which solves

$$\underset{x\in N}{\text{minimize}}\ \sigma(x)$$

i.e., the point which solves the *minimax* problem

$$\underset{x\in N}{\text{minimize}} \left\{ \underset{j\in N}{\text{maximum}}\ d(x,j) \right\}$$

The *Edge Center* of an edge [J,K] is the point z on edge [J,K] which solves

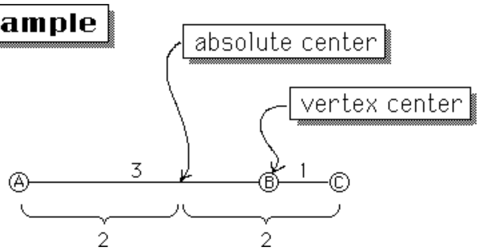
$$\underset{x\in [J,K]}{\text{minimize}}\ \sigma(x)$$

The *Absolute Center* of a network is the point z (a node or a point on an edge) which solves

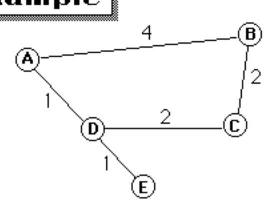
$$\underset{x\in G}{\text{minimize}}\ \sigma(x)$$

where $G = N \cup A$ is the set of nodes and points on edges in the edge set A

Example



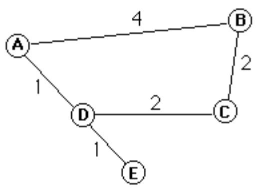
Example



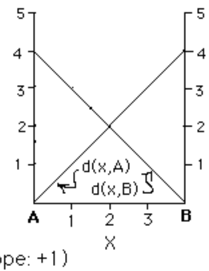
Where should a fire station be located so as to minimize the distance to the farthest village?

d(x,J) = shortest path from point x (on the network) to village J, J∈N ={A,B,C,D,E}

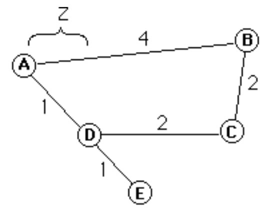
$$\underset{X}{\text{Minimize}} \{ \underset{J\in N}{\text{Max}}\ d(x,J) \}$$

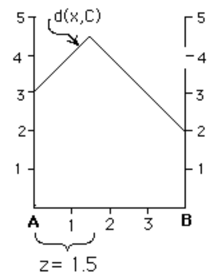


Consider d(x,J) for points x on the edge (A,B)



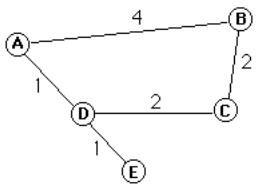
d(x,A) is monotonically increasing (slope: +1) as x moves from A to B, while d(x,B) is monotonically decreasing (slope: -1)



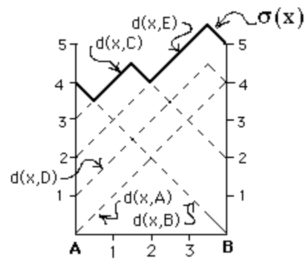


d(x,C) = 3 at x=A, and increases (slope= +1) as x moves toward B. At the point x where d(x,A)+1+2 = d(x,B)+2, the function begins to decrease (slope= -1).

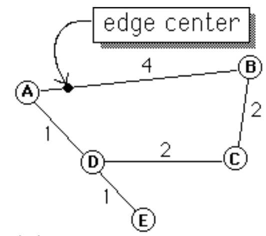
$z+1+2=(4-z)+2 \implies z=1.5$



$\sigma(x) = \underset{j \in N}{\text{maximum}}\ d(x,j)$

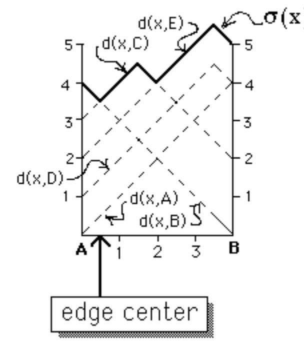


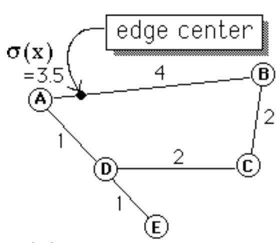
$\sigma(x)$ is the upper envelope of the family of functions d(x,J), J∈N



$\sigma(x) = \underset{j \in N}{\text{maximum}}\ d(x,j)$

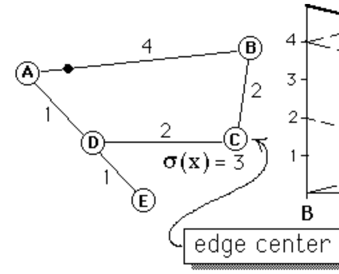
The point which minimizes the function σ on [A,B] lies 0.5 miles from A

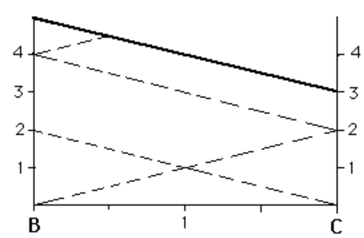




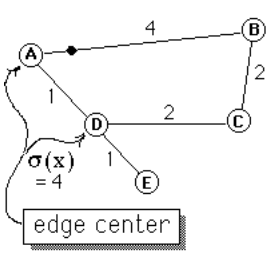
$\sigma(x) = \underset{j \in N}{\text{maximum}}\ d(x,j)$

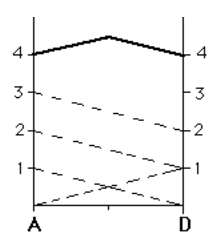
The absolute center may be found by computing each edge center, and selecting the best.



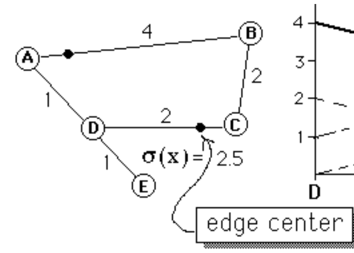


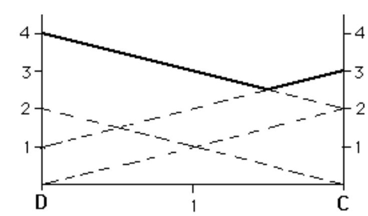
$\sigma(x) = \underset{j \in N}{\text{maximum}}\ d(x,j)$



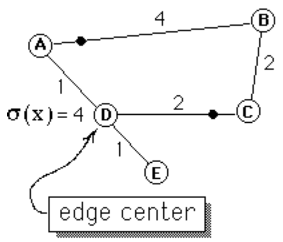


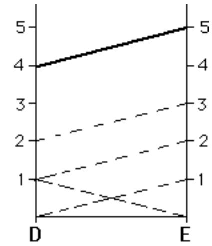
$\sigma(x) = \underset{j \in N}{\text{maximum}}\ d(x,j)$





$\sigma(x) = \underset{j \in N}{\text{maximum}}\ d(x,j)$





$$\sigma(x) = \underset{j \in N}{\text{maximum}}\ d(x,j)$$

edge	edge center	σ(x)
[A,B]	0.5 from A	3.5
[B,C]	Vertex C	3
[C,D]	0.5 from C	2.5
[A,D]	Vertices A&D	4
[D,E]	Vertex D	4

absolute center of network (2.5)

Searching some edges for their centers may be avoided by using the lower bound provided by

Theorem Let X_{pq} be the edge center of [P,Q].

Then

$$\sigma(X_{pq}) \geq \frac{\sigma(P) + \sigma(Q) - d(P,Q)}{2}$$

If this lower bound exceeds σ at the vertex center of the network, then the absolute center cannot lie on this edge!

Proof

Proof:

$d(X,j) \leq \sigma(X) \quad \forall j$

$d(P,j) \leq d(P,X) + d(X,j)$

$d(P,j) \leq d(P,X) + \sigma(X)$

But $\sigma(P) = \max_j \{ d(P,j) \}$,

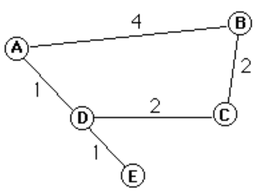
$\Rightarrow \sigma(P) \leq d(P,X) + \sigma(X)$

Likewise, $\sigma(Q) \leq d(Q,X) + \sigma(X)$

Sum these two inequalities:

$$\sigma(P) + \sigma(Q) \leq 2\,\sigma(X) + \underbrace{d(P,X) + d(X,Q)}_{d(P,Q)}$$

$$\Rightarrow \sigma(X) \geq \frac{\sigma(P) + \sigma(Q) - d(P,Q)}{2}$$



vertex	σ
A	4
B	5
C	3
D	4
E	5

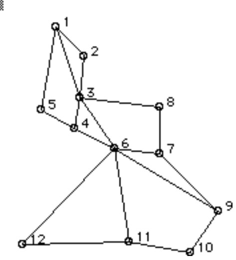
vertex center (C)

edge	lower bound	min σ(X)
[A,B]	2.5	3.5
[B,C]	3	3
[C,D]	2.5	2.5
[A,D]	3.5	4
[D,E]	4	4

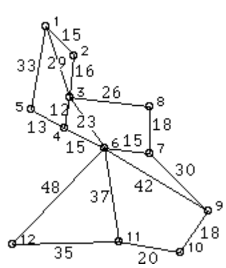
Using the lower bound would have eliminated 3 edges from consideration!

The edge centers needed to be found only for [A,B] & [C,D]

Example



Example



Shortest Path Lengths

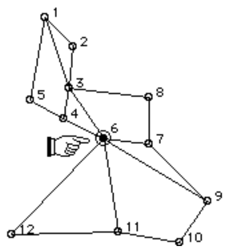
from \ to	1	2	3	4	5	6	7	8	9	0	1	2	σ(X)
1	0	15	29	41	33	52	67	55	94	109	89	100	109
2	15	0	16	28	41	39	54	42	81	96	76	87	96
3	29	16	0	12	25	23	38	26	65	80	60	71	80
4	41	28	12	0	13	15	30	38	57	72	52	63	72
5	33	41	25	13	0	28	43	51	70	85	65	76	85
6	52	39	23	15	28	0	15	33	42	57	37	48	57
7	67	54	38	30	43	15	0	18	30	48	52	63	67
8	55	42	26	38	51	33	18	0	48	66	70	81	81
9	94	81	65	57	70	42	30	48	0	18	38	73	94
10	109	96	80	72	85	57	48	66	18	0	20	55	109
11	89	76	60	52	65	37	52	70	38	20	0	35	89
12	100	87	71	63	76	48	63	81	73	55	35	0	100

vertex center (57)

Vertex Center of Network

(Which minimizes the maximum distance to farthest nodes)

Vertex center of the network is at node 6
where maximum distance (to node 10) is 57



Lower Bound of σ on the edges

i	j	LB	
6	7	54.5	
6	9	54.5	
6	11	54.5	
6	12	54.5	
3	6	57	X
4	6	57	X
7	8	65	X
7	9	65.5	X
3	8	67.5	X
3	4	70	X
4	5	72	X
11	12	77	X
1	3	80	X
2	3	80	X
1	5	80.5	X
10	11	89	X
9	10	92.5	X
1	2	95	X

eliminated by L.B. (rows 3–6 through 1–2)

Only 4 edges could have edge centers better than the vertex center where σ =57

The function σ on edge [6,7]

Monotonically increasing distance functions: d(x,k) where

k=	1	2	3	4	5	6	11	12
d(i,k)=	52	39	23	15	28	0	37	48
d(j,k)=	67	54	38	30	43	15	52	63

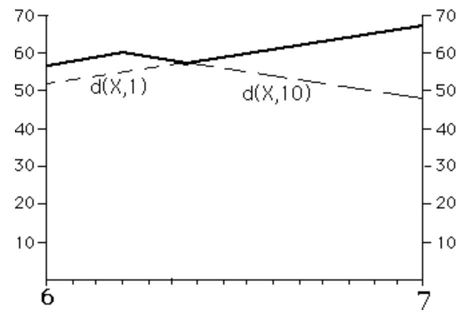
Monotonically decreasing distance functions: d(x,k) where

k=	7	8
d(i,k)=	15	33
d(j,k)=	0	18

Distance functions which increase to a peak at a point Δ units from i, then decrease: d(x,k) where

k=	9	10
d(i,k)=	42	57
d(j,k)=	30	48
Δ=	1.5	3

The function σ on edge [6,7]



The edge center is at vertex #6

The function σ on edge [6,9]

Monotonically increasing distance functions: d(x,k) where

k=	1	2	3	4	5	6
d(i,k)=	52	39	23	15	28	0
d(j,k)=	94	81	65	57	70	42

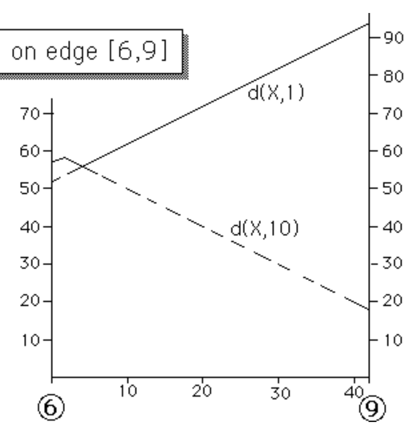
Monotonically decreasing distance functions: d(x,k) where

k=	9
d(i,k)=	42
d(j,k)=	0

Distance functions which increase to a peak at a point Δ units from i, then decrease: d(x,k) where

k=	7	8	10	11	12
d(i,k)=	15	33	57	37	48
d(j,k)=	30	48	18	38	73
Δ=	28.5	28.5	1.5	21.5	33.5

σ on edge [6,9]



The edge center is 4 units from vertex 6, with σ(X) = 56

The function σ on edge [6,11]

Monotonically increasing distance functions: d(x,k) where

k=	1	2	3	4	5	6	7	8
d(i,k)=	52	39	23	15	28	0	15	33
d(j,k)=	89	76	60	52	65	37	52	70

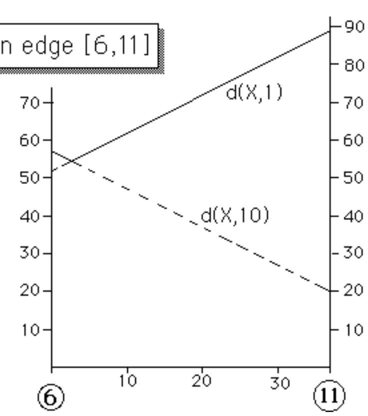
Monotonically decreasing distance functions: d(x,k) where

k=	10	11
d(i,k)=	57	37
d(j,k)=	20	0

Distance functions which increase to a peak at a point Δ units from i, then decrease: d(x,k) where

k=	9	12
d(i,k)=	42	48
d(j,k)=	38	35
Δ=	16.5	12

σ on edge [6,11]



The edge center is 2.5 units from vertex #6, with σ(X)= 54.5

The function σ on edge [6,12]

Monotonically increasing distance functions: d(x,k) where

k=	1	2	3	4	5	6	7	8
d(i,k)=	52	39	23	15	28	0	15	33
d(j,k)=	100	87	71	63	76	48	63	81

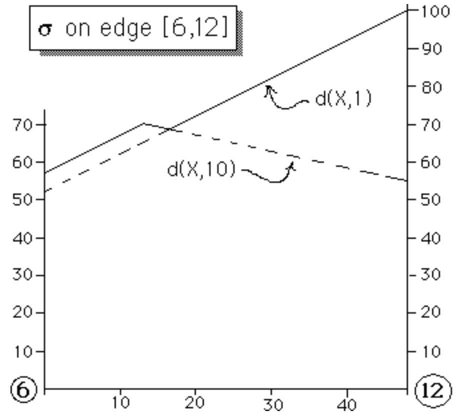
Monotonically decreasing distance functions: d(x,k) where

k=	12
d(i,k)=	48
d(j,k)=	0

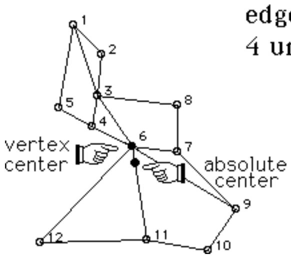
Distance functions which increase to a peak at a point Δ units from i, then decrease: d(x,k) where

k=	9	10	11
d(i,k)=	42	57	37
d(j,k)=	73	55	35
Δ=	39.5	23	23

σ on edge [6,12]



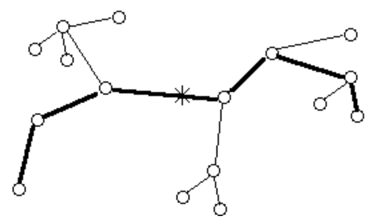
The edge center is at vertex #6



The absolute center is the edge center of edge [6,11], 4 units from vertex #6, with σ(X*)=54.5

Center of a Tree

A center of a tree lies at the midpoint of the longest elementary chain in the tree.

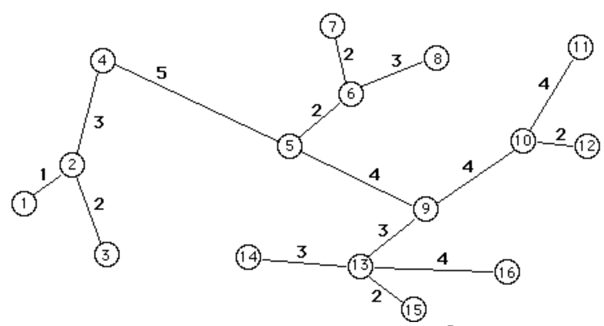


Finding Center of a Tree

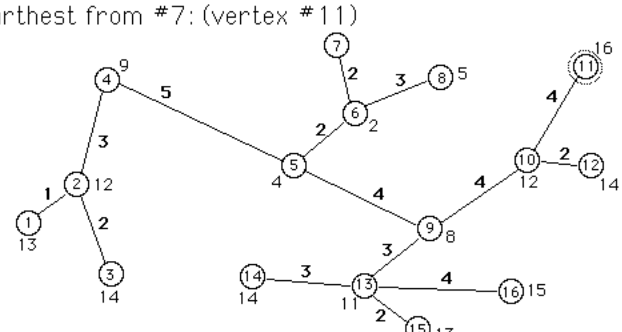
0. Choose arbitrarily a point X of the tree.
1. Find the vertex *farthest* from X. Call this vertex V_1. (This will have degree 1.)
2. Find the vertex *farthest* from V_1. Call this vertex V_2. (This will also have degree 1.)
3. Find the midpoint X* of the unique elementary path from V_1 to V_2. X* will be the *absolute* center of the tree, and the vertex nearest to X* will be the *vertex* center.

Example

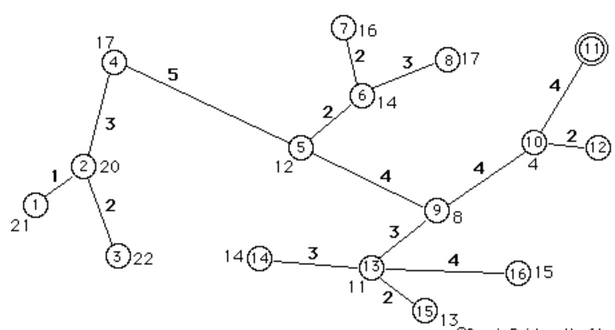
Find the absolute & vertex centers of the tree:



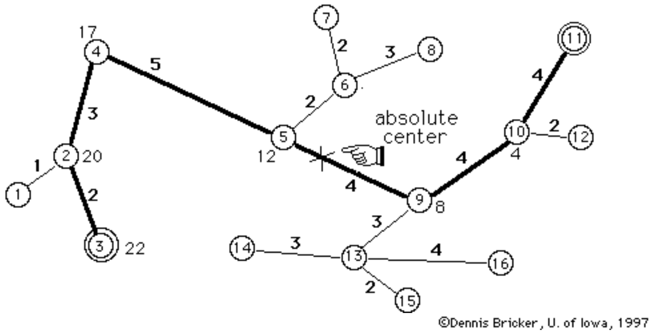
Arbitrarily choose vertex 7. Label each vertex with its distance from vertex 7, to find that farthest from #7: (vertex #11)



Now label the vertices with their distances from vertex #11, to find that farthest from #11: vertex 3.



The midpoint of the chain from vertex 11 to vertex 3
is a distance 11 from vertex 11, on the edge [5,9]



The vertex center of the tree is at vertex #5,
the vertex nearest to the absolute center.