Balas' Additive Algorithm

Implicit Enumeration for 0-1 Integer LP

author

Egon Balas' algorithm for optimally solving zero-one LP problems is often referred to as...

Implicit Enumeration

and, because it requires only addition & subtraction (no multiplication or divisions),

Additive Algorithm

Standard Form

Let's assume that the problem is of the form:

$$\text{Minimize } z = \sum_{j\in N} C_j X_j$$

$$\text{subject to } \sum_{j\in N} a_{ij} X_j \le b_i, \ \forall i \in M$$

$$X_j \in \{0,1\}, \ \forall j \in N$$

where $M=\{1,2,3,\ldots,m\}$ and $N=\{1,2,3,\ldots,n\}$

and $C_j \ge 0 \ \forall j \in N$

nonnegative costs!

Example

Maximize $-2X_1 + X_2 - 3X_3 + X_4$

subject to

$$X_1 + 2X_2 - X_3 \ge 1$$

$$-2X_1 + X_2 - X_4 \le 3$$

$$X_j \in \{0,1\}, \ j=1,2,3,4$$

NOT in standard form...

objective is maximize, not minimize
costs differ in sign
one constraint is "greater-than-or-equal"

Replace "Max z" with "- Min -z" and "≥" with "≤"

$-$ Minimize $2X_1 - X_2 + 3X_3 - X_4$

subject to

$$-X_1 - 2X_2 + X_3 \le -1$$

$$-2X_1 + X_2 - X_4 \le 3$$

$$X_j \in \{0,1\}, \ j=1,2,3,4$$

For each variable X_j *having a negative cost, substitute* $1 - Y_j$ *where* $Y_j \in \{0,1\}$ *is the complement of* X_j.

$-$ Minimize $2X_1 - (1-Y_2) + 3X_3 - (1-Y_4)$

subject to

$$-X_1 - 2(1-Y_2) + X_3 \le -1$$

$$-2X_1 + (1-Y_2) - (1-Y_4) \le 3$$

$$X_j \in \{0,1\}, \ j=1,3$$

$$Y_j \in \{0,1\}, \ j=2,4$$

That is, the original problem is equivalent to the following problem, which is in the "standard form" for Balas' algorithm:

$2 -$ Minimize $2X_1 + Y_2 + 3X_3 + Y_4$

subject to

$$-X_1 + Y_2 + X_3 \le 1$$

$$-2X_1 - Y_2 + Y_4 \le 2$$

$$X_j \in \{0,1\}, \ j=1,3$$

$$Y_j \in \{0,1\}, \ j=2,4$$

Example

Minimize $3X_1 + 8X_2 + X_3 + 16X_4 + X_5$

subject to
$$X_1 - 2X_2 - 6X_3 + 2X_4 + 3X_5 \le 0$$
$$X_1 - 3X_3 - 2X_4 + 2X_5 \le -2$$
$$X_1 - 5X_2 + 4X_3 - X_4 - 2X_5 \le -5$$
$$X_j \in \{0,1\},\ j=1,2,3,4,5$$

There are $2^5 = 32$ binary vectors of length 5, which we could explicitly enumerate.

Minimize $3X_1 + 8X_2 + X_3 + 16X_4 + X_5$

subject to
$$X_1 - 2X_2 - 6X_3 + 2X_4 + 3X_5 \le 0$$
$$X_1 - 3X_3 - 2X_4 + 2X_5 \le -2$$
$$X_1 - 5X_2 + 4X_3 - X_4 - 2X_5 \le -5$$
$$X_j \in \{0,1\},\ j=1,2,3,4,5$$

For each of the 32 binary vectors, let's evaluate

$$\begin{cases} z = 3X_1 + 8X_2 + X_3 + 16X_4 + X_5 \\ g_1(X) = X_1 - 2X_2 - 6X_3 + 2X_4 + 3X_5 \le 0 \\ g_2(X) = X_1 - 3X_3 - 2X_4 + 2X_5 \le -2 \\ g_3(X) = X_1 - 5X_2 + 4X_3 - X_4 - 2X_5 \le -5 \end{cases}$$

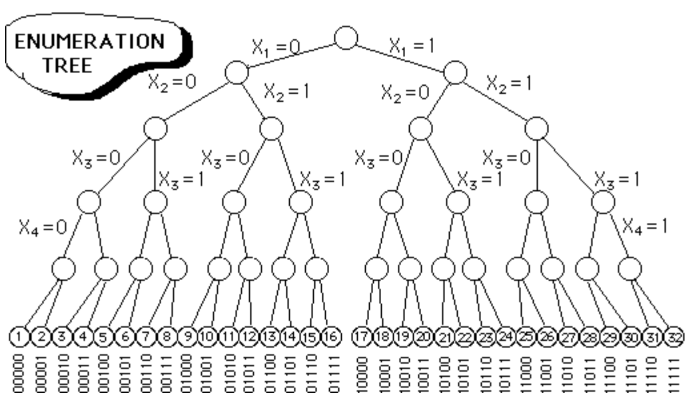
#	X	z	g_1	g_2	g_3
1	0 0 0 0 0	0	0	0	0
2	0 0 0 0 1	1	3	2	-2
3	0 0 0 1 0	16	2	-2	-1
4	0 0 0 1 1	17	5	0	-3
5	0 0 1 0 0	1	-6	-3	4
6	0 0 1 0 1	2	-3	-1	2
7	0 0 1 1 0	17	-4	-5	3
8	0 0 1 1 1	18	-1	-3	1
9	0 1 0 0 0	8	-2	0	-5
10	0 1 0 0 1	9	1	2	-7
11	0 1 0 1 0	24	0	-2	-6
12	0 1 0 1 1	25	3	0	-8
13	0 1 1 0 0	9	-8	-3	-1
14	0 1 1 0 1	10	-5	-1	-3
15	0 1 1 1 0	25	-6	-5	-2
16	0 1 1 1 1	26	-3	-3	-4

#	X	z	g_1	g_2	g_3
17	1 0 0 0 0	3	1	1	1
18	1 0 0 0 1	4	4	3	-1
19	1 0 0 1 0	19	3	-1	0
20	1 0 0 1 1	20	6	1	-2
21	1 0 1 0 0	4	-5	-2	5
22	1 0 1 0 1	5	-2	0	3
23	1 0 1 1 0	20	-3	-4	4
24	1 0 1 1 1	21	0	-2	2
25	1 1 0 0 0	11	-1	1	-4
26	1 1 0 0 1	12	2	3	-6
27	1 1 0 1 0	27	1	-1	-5
28	1 1 0 1 1	28	4	1	-7
29	1 1 1 0 0	12	-7	-2	0
30	1 1 1 0 1	13	-4	0	-2
31	1 1 1 1 0	25	-5	-4	-1
32	1 1 1 1 1	26	-2	-2	-3

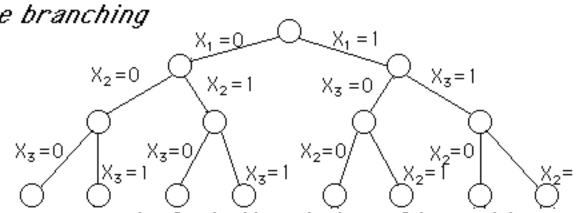
#	X	z	g_1	g_2	g_3
1	0 0 0 0 0	0	0	0	0
2	0 0 0 0 1	1	3	2	-2
3	0 0 0 1 0	16	2	-2	-1
4	0 0 0 1 1	17	5	0	-3
5	0 0 1 0 0	1	-6	-3	4
6	0 0 1 0 1	2	-3	-1	2
7	0 0 1 1 0	17	-4	-5	3
8	0 0 1 1 1	18	-1	-3	1
9	0 1 0 0 0	8	-2	0	-5
10	0 1 0 0 1	9	1	2	-7
11	0 1 0 1 0	24	0	-2	-6
12	0 1 0 1 1	25	3	0	-8
13	0 1 1 0 0	9	-8	-3	-1
14	0 1 1 0 1	10	-5	-1	-3
15	0 1 1 1 0	25	-6	-5	-2
16	0 1 1 1 1	26	-3	-3	-4

#	X	z	g_1	g_2	g_3
17	1 0 0 0 0	3	1	1	1
18	1 0 0 0 1	4	4	3	-1
19	1 0 0 1 0	19	3	-1	0
20	1 0 0 1 1	20	6	1	-2
21	1 0 1 0 0	4	-5	-2	5
22	1 0 1 0 1	5	-2	0	3
23	1 0 1 1 0	20	-3	-4	4
24	1 0 1 1 1	21	0	-2	2
25	1 1 0 0 0	11	-1	1	-4
26	1 1 0 0 1	12	2	3	-6
27	1 1 0 1 0	27	1	-1	-5
28	1 1 0 1 1	28	4	1	-7
29	1 1 1 0 0	12	-7	-2	0
30	1 1 1 0 1	13	-4	0	-2
31	1 1 1 1 0	25	-5	-4	-1
32	1 1 1 1 1	26	-2	-2	-3

Solution #11 is the only one feasible in all 3 constraints



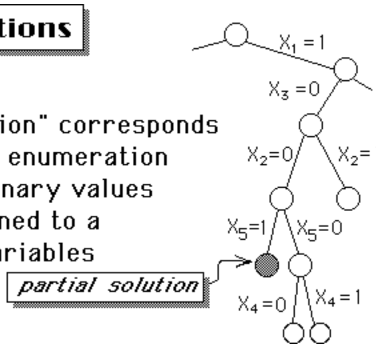
The order of branching is not important, e.g., one can branch on X_3 before branching on X_2



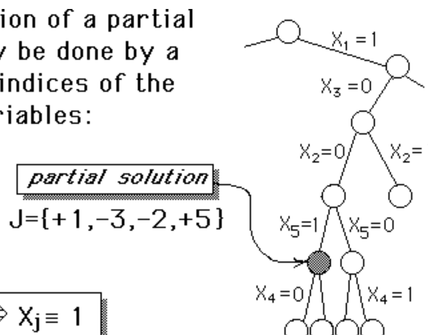
In fact, the choice of branching variable may differ on the same level of the tree!

Partial Solutions

A "partial solution" corresponds to a node of the enumeration tree in which binary values have been assigned to a subset of the variables



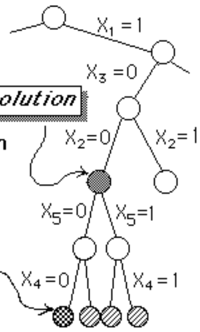
Representation of a partial solution may be done by a vector of ± indices of the assigned variables:



$\{\ldots,+j,\ldots\} \Rightarrow X_j \equiv 1$

$\{\ldots,-j,\ldots\} \Rightarrow X_j \equiv 0$

Completions



partial solution

The completions of a partial solution consist of ALL of the nodes at the bottom-most level of the tree, where all variables have been assigned.

zero completion

The completion with all free variables assigned value of zero is the "zero completion"

completions of partial solution

Fathoming of a Partial Solution

A partial solution (node) of an enumeration tree may be considered fathomed if one of the following may be demonstrated:

- all completions violate one or more constraints
- all completions are inferior (with respect to the objective) to the incumbent
- the zero completion is feasible & superior to the incumbent (& therefore becomes the new incumbent)

Fathoming Test #1

A free variable X_j ($j \notin J$) which has nonnegative coefficients in *every* constraint which is violated by the zero completion should be zero, since assigning it the value 1 will improve neither the objective function nor feasibility.

Compute

$$A = \{ j \mid j \in N - J, a_{ij} \geq 0 \ \forall \ i \in M \text{ such that } S_i < 0 \}$$

and

$$N^1 = N - J - A$$

indices of free variables which are eligible to be assigned value 1

If $N^1 = \varnothing$, then the partial solution J may be fathomed!

FATHOMING TEST ONE

Fathoming Test #2

Let Z be the objective function value of the zero completion of the partial solution J.

If $Z + C_k \geq \underline{Z}$ (the incumbent) for some $k \notin J$, then no completion of J which has $X_k = 1$ can be optimal!

Compute

$$B = \{ j \mid j \in N^1, Z + C_j \geq \underline{Z} \}$$

and

$$N^2 = N^1 - B$$

indices of all free variables which are eligible to be assigned value 1

If $N^2 = \varnothing$, then the partial solution may be fathomed!

FATHOMING TEST TWO

Fathoming Test #3

If constraint #i is violated by the zero completion of the partial solution, so that the slack $S_i < 0$,

and if the sum of all negative coefficients of the free variables (in N^2) exceeds S_i,

Then no feasible completion of the partial solution exists.

Compute

$$C = \left\{ i \;\middle|\; S_i < \sum_{j \in N^2} a_{ij}^- \right\}$$

If $C \neq \varnothing$ then the partial solution is fathomed.

FATHOMING TEST THREE

Selection of a Free Variable for Forward Step

When the fathoming tests fail to fathom the current partial solution, branching will be performed, by fixing a free variable X_j

$J \leftarrow J,\{+j\}$

The positive index "j" is appended to the end of the current J vector

Any free variable might be chosen.... is there a "best" choice?

Let S_i = slack in constraint #i in the zero completion of J

Then $S_i - a_{ij}$ = slack in constraint #i if free variable X_j=1 while other free variables are assigned value zero

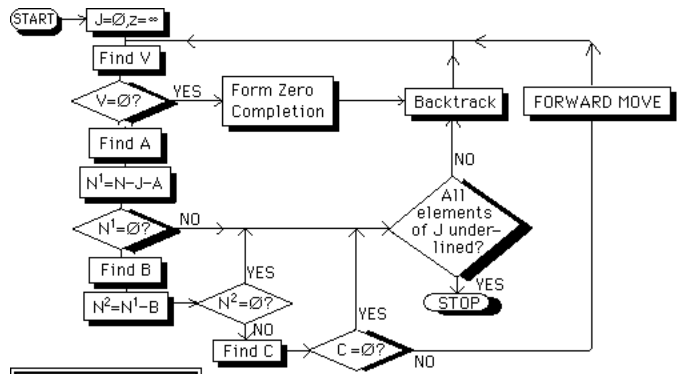
Define $(S_i - a_{ij})^- = \min\{0, S_i - a_{ij}\}$ *NEGATIVE PART*

$v_j = \sum_i (S_i - a_{ij})^-$ measures the infeasibility which results from fixing X_j=1

Balas' strategy was to choose the free variable which would result in the *least* infeasibility, i.e., the maximum ("least negative") value of v_j

$$j^* = \underset{j \in N^2}{\text{argmax}} \{v_j\} = \underset{j \in N^2}{\text{argmax}} \sum_i (S_i - a_{ij})^-$$

Other rules might result in partial solutions which are more easily fathomed.



Flowchart

Minimize $4X_1 + 8X_2 + 9X_3 + 3X_4 + 4X_5 + 10X_6$

s.t.

$$\begin{cases} 4X_1 - 5X_2 - 3X_3 - 2X_4 - X_5 + 8X_6 \le -8 \\ -5X_1 + 2X_2 + 9X_3 + 8X_4 - 3X_5 + 8X_6 \le 7 \\ 8X_1 + 5X_2 - 4X_3 + X_5 + 6X_6 \le 6 \end{cases}$$

$$X_j \in \{0, 1\} \ \forall j = 1, \ldots, 6$$

Random ILP (seed = 148458)

\# variables = 6
\# constraints = 3

1	2	3	4	5	6		b
4	8	9	3	4	10		min
4	-5	-3	-2	-1	8	≤	-8
-5	2	9	8	-3	8	≤	7
8	5	-4	0	1	6	≤	6

Constraints are of the form Ax≤b

Inserting slack variables:

$$4X_1 - 5X_2 - 3X_3 - 2X_4 - X_5 + 8X_6 + S_1 = -8$$
$$-5X_1 + 2X_2 + 9X_3 + 8X_4 - 3X_5 + 8X_6 + S_2 = 7$$
$$8X_1 + 5X_2 - 4X_3 + X_5 + 6X_6 + S_3 = 6$$

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 6	2 3 4 5		2 3 4 5		-3 -7 -7 -7	2	***

(1) $J = \varnothing$

Constraints violated by zero completion:

$S_1 = -8$ ← violation!
$S_2 = 7$ ok
$S_3 = 6$ ok

A = {1,6}: variables which cannot improve feasibility in violated constraints if equal to 1

$$4X_1 - 5X_2 - 3X_3 - 2X_4 - X_5 + 8X_6 + S_1 = -8$$

Constraint #1

nonnegative coefficients in violated constraint!

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 6	2 3 4 5		2 3 4 5		-3 -7 -7 -7	2	***

(1) $J = \varnothing$

$N^1 = N - J - A = \{1,2,3,4,5,6\} - \varnothing - \{1,6\} = \{2,3,4,5\}$

Indices of free variables which might be assigned value of 1

FATHOMING TEST #1

$N^1 \neq \varnothing$, so this test fails to fathom the partial solution!

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 6	2 3 4 5		2 3 4 5		-3 -7 -7 -7	2	***

(1) $J = \varnothing$ Fathoming Test #2 isn't applicable, since we do not yet have a finite incumbent.

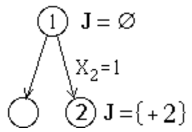
$$4 X_1 - 5 X_2 - 3 X_3 - 2 X_4 - X_5 + 8 X_6 + S_1 = -8$$

It is possible to satisfy constraint #1 by assigning values to the free variables having negative coefficients, e.g.,

$X_2 = X_3 = X_4 = X_5 = 1 \Rightarrow S_1 = -8 + 5 + 3 + 2 + 1 = 3 > 0$ feasible! $\Rightarrow C = \varnothing$

FATHOMING TEST #3 This test fails to fathom the partial sol'n

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 6	2 3 4 5		2 3 4 5		-3 -7 -7 -7	2	***
2	2	1	1 6	3 4 5		3 4 5		-4 -4 -2	5	***



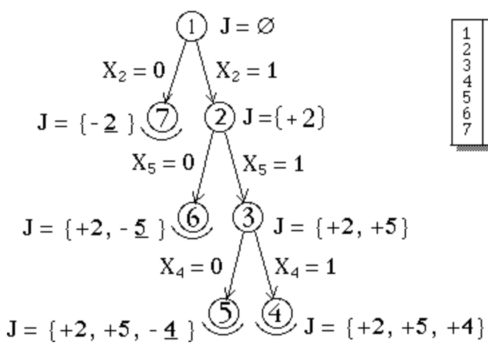
	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 6	2 3 4 5		2 3 4 5		-3 -7 -7 -7	2	***

(1) $J = \varnothing$ Since the fathoming tests have all failed, we must next choose a variable for branching.

Variable	constraint infeasibility if =1			Total
	1	2	3	
2	-3	0	0	-3
3	-5	-2	0	-7
4	-6	-1	0	-7
5	-7	0	0	-7

Least amount of infeasibility if assigned 1

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 6	2 3 4 5		2 3 4 5		-3 -7 -7 -7	2	***
2	2	1	1 6	3 4 5		3 4 5		-4 -4 -2	5	***
3	2 5	1	1 6	3 4		3 4		-1	4	***
4	2 5 4									***
5	2 5 -4	1	1 6	3	3					15
6	2 -5	1	1 6	3 4	3	4	1			15
7	-2	1	1 6	3 4 5		3 4 5	1			15



	J
1	
2	2
3	2 5
4	2 5 4
5	2 5 -4
6	2 -5
7	-2

Random ILP (seed = 148458)

Solution is:

i	1	2	3	4	5	6
X[i]	0	1	0	1	1	0

Objective function value is 15

Example Problem

\# variables = 5
\# constraints = 3

1	2	3	4	5		b
5	7	10	3	1		min
-1	3	-5	-1	4	≤	-2
2	-6	3	2	-2	≤	0
0	1	-2	1	1	≤	-1

Constraints are of the form Ax≤b

iteration	J	V1	A	N1	B	N2	C	v	j	Z*
1		1 3	2 5	1 3 4	***	1 3 4		-4 -3 -5	3	***
2	3	2	1 4	2 5	***	2 5		-2	2	***
3	3 2				***					***
4	3 -2	2	1 4	5	***	5	2			17
5	-3	1 3	2 5	1 4	***	1 4	3			17

Balas' Additive Algorithm

Example Problem

CPU time= 1.75 sec.

Solution is:

i	1	2	3	4	5
X[i]	0	1	1	0	0

Objective function value is 17

Random ILP (seed = 825025)

variables = 8
constraints = 5

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

Constraints are of the form Ax≤b

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 3 5 7	2 4 6 8		2 4 6 8		-3 -1 0 -4	6	***
2	6									***
3	-6	1	1 3 5 7	2 4 8	4	2 8		-3 -4	2	9
4	-6 2	2 4	3 7 8	1 4 5	4 5	1	2			9
5	-6 -2	1	1 3 5 7	4 8	4	8		-4	8	9
6	-6 -2 8	4	3 5 7	1 4	1 4					9
7	-6 -2 -8	1	1 3 5 7	4	4					9

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 3 5 7	2 4 6 8		2 4 6 8		-3 -1 0 -4	6	***

The first constraint is violated by the zero completion (S =-2).
Variables 1,3,5, &7 have positive coefficients in this constraint, and thus cannot help in achieving feasibility. They form the set A, which are implicitly fixed = 0, leaving N = {2, 4, 6, 8}.
Test 2 isn't applicable because no incumbent has been identified.
Test 3 considers the violated constraints in V1 to determine whether it is possible to satisfy them. In this case, we see that increasing any one of variables 2,4,6, or 8 will result in feasibility, so C is empty.
The fathoming tests have failed, and therefore we must perform a forward branch.

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

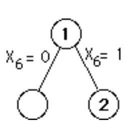
	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 3 5 7	2 4 6 8		2 4 6 8		-3 -1 0 -4	6	***

Choosing the branching variable:
Setting variable 2 equal to 1 results in constraint violations {0, 1, 0, 2, 0} and so V2 = -3.
Setting variable 4 equal to 1 results in constraint violations {0, 1, 0, 0, 0} and so V4 = -1
Setting variable 6 equal to 1 results in constraint violations {0, 0, 0, 0, 0} and so V6=0.
Setting variable 8 equal to 1 results in constraint violations {0, 0, 0, 4, 0} and so V8 = 0.

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
1		1	1 3 5 7	2 4 6 8		2 4 6 8		-3 -1 0 -4	6	***

The (rather arbitrary) rule is to select that variable causing the least infeasibility, and so variable 6 is selected for the branching.
Therefore, J, which was previously empty, is now {+6}.

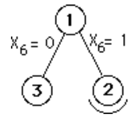


1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
2	6									***

At node 2, J={+6} and no constraints are violated by the zero completion (i.e., X = 1 and all other variables zero).
Since no other completion of this partial solution can cost less than the zero completion, the node is fathomed, and we may backtrack.

Backtracking: J becomes {-6}



1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

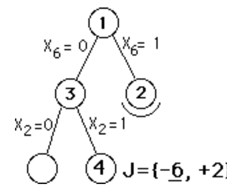
	J	V1	A	N1	B	N2	C	v	j	Z*
3	-6	1	1 3 5 7	2 4 8	4	2 8		-3 -4	2	9

At node 3, again only the first constraint is violated by the zero completion, and variables 1, 3, 5, & 7 cannot contribute toward making this constraint feasible, so that they are implicitly fixed at value zero, leaving only free variables 2, 4, & 8.
If X2 or X8 were fixed at value 1, the objective function is less than the incumbent, but if X4 were fixed at 1, the objective function would exceed the incumbent (B = {4}) and therefore is implicitly fixed at value 0, leaving only N = {2, 8} as free variables. Fixing either of these at value 1 would satisfy the violated constraint (#1), so C is empty.

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
3	-6	1	1 3 5 7	2 4 8	4	2 8		-3 -4	2	9

Therefore we cannot fathom this node, and must make a forward move, i.e., branch.
Selection of branching variable: Fixing variable 2 at 1 gives constraint violations 0, 0, 1, 0, 2, 0, while fixing variable 8 at 1 gives violations 0, 0, 0, 4, 0. Variable 2 results in less infeasibility, and is selected for branching.



1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

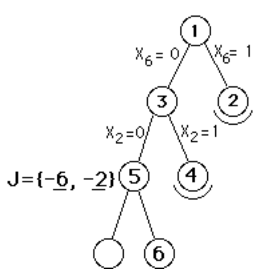
	J	V1	A	N1	B	N2	C	v	j	Z*
4	-6 2	2 4	3 7 8	1 4 5	4 5	1	2			9

At node 4, constraints 2 & 4 are violated by the zero completion, but variables 3, 7, & 8 cannot assist in making these constraints feasible, and are therefore implicitly set equal to zero, leaving variables 1, 4, &5 as free variables.
Consider X4: together with X2 this gives a cost of 13, exceeding the incumbent (9); likewise, variable X5 together with X2 gives a cost of 9 which is no better than the incumbent. Hence variables 4&5 may be implicitly fixed at value zero, leaving only variable 1 as a free variable.

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
4	-6 2	2 4	3 7 8	1 4 5	4 5	1	2			9

With variable 2 equal to 1 and only variable 1 free, we can determine that the violated constraint #2 cannot be made feasible. (Constraint 4 could be made feasible by setting X1 = 1.) Hence C={2} and the subproblem is fathomed.
We must now backtrack:
Currently J = {-6, +2} and so the next node will have J={+6, -2}.



1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
5	-6 -2	1	1 3 5 7	4 8	4	8		-4	8	9

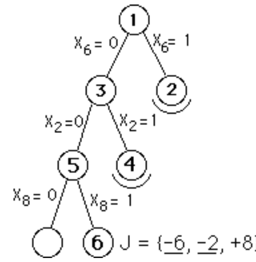
At node 5, variables 2 & 6 are zero, and again constraint 1 is violated by the zero completion.
Variables 1, 3, 5, & 7 cannot help to achieve feasibility of this constraint (since they have positive coefficients) and therefore they can be made implicitly zero, leaving only variables 4 & 8 as free variables.
Variable 4, if set = 1, would cause the cost to exceed the incumbent, and therefore is implicitly fixed at zero, leaving only variable 8 free

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
5	-6 -2	1	1 3 5 7	4 8	4	8		-4	8	9

We see that with only variable 8, it is possible to satisfy constraint 1 (by setting X8 = 1), so C is empty.
Fixing X8=1 results in infeasibilities 0, 0, 0, 4, 0. Obviously variable 8 is chosen for the branching.

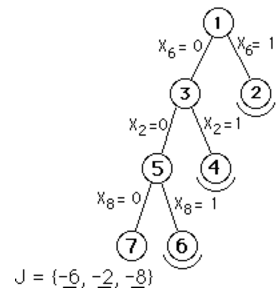
J, which was {-6, -2}, is extended on the right by +8, i.e.,
$J = \{-\underline{6}, -\underline{2}, +8\}$.



1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
6	-6 -2 8	4	3 5 7	1 4	1 4					9

At node 6, the zero completion violates constraint 4, and the free variables 3, 5, & 7 cannot help to remove the feasibility, and hence are implicity fixed at value zero, leaving only variables 1 & 4 as free variables.
However, increasing variable 1 would result in a cost of 6+3, which is no better than the incumbent, while increasing variable 4 would result in a cost of 15, worse than the incumbent. These two variables are implicitly fixed at value zero, therefore, leaving no free variables. The node is fathomed.

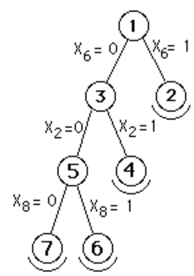


To backtrack from J={-6, -2, +8}, we look for the last element without underline, reverse its sign, and underline it, giving us

1	2	3	4	5	6	7	8		b
3	4	5	9	5	9	4	6		min
3	-5	4	-2	6	-4	6	-5	≤	-2
0	8	1	8	-2	2	0	4	≤	7
9	2	4	7	-3	2	6	1	≤	16
-5	2	5	-2	6	-4	0	4	≤	0
9	-1	1	1	-3	6	7	0	≤	16

	J	V1	A	N1	B	N2	C	v	j	Z*
7	-6 -2 -8	1	1 3 5 7	4	4					9

At node 7, variables 2, 6, &8 are all fixed at zero, and the first constraint is violated by the zero completion. Variables 1, 3, 5, and 7 all have positive coefficients in this constraint and are therefore unable to assist in gaining feasibility. Hence they are implicitly fixed at value zero, leaving only variable 4 as a free variable. However, setting variable 4 equal to 1 gives a cost (9) which is no better than the incumbent, and therefore this node can be fathomed.



To backtrack, we look for the right-most element without underline. there are none, and therefore the tree is fathomed.
$J = \{-\underline{6}, -\underline{2}, -\underline{8}\}$

The current incumbent is therefore optimal.

That is, $X_j = 0$ except for j=6 (found at node 2.)