DP Model: Option Pricing

An *American Put Option* gives the holder

the *right* (but not the obligation)
to *sell* a specified quantity of a commodity
at a specified “strike” *price*
at *any time* the holder chooses,
on or before a specified *expiration* date.
(A European Put Option can only be exercised at the expiration date.)

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EXAMPLE:

- The *current price* of a share of stock in XYZ is P = $2.00.
- You have an option to sell 100 shares of this stock for P' = $2.10 (*"strike price"*) at any time you choose within the next six months.
- Assume the time value of money is 5% per annum.
- The annual *volatility* of the commodity price is σ=0.2.

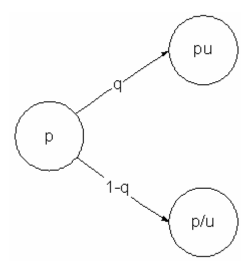
Volatility: A measure of the previous fluctuations in share price (crudely: an indicator of the commodity's up/downess). Usually, the standard deviation of the log of price returns.
--http://www.numa.com/ref/volatili.htm

What is the expected value of this option?

*(Clearly it is worth **at least $10**, because it could be exercised immediately by buying 100 shares at $2 and selling them for $2.10 each.)*

ASSUMPTIONS

We assume the *Cox-Ross-Rubenstein* binomial option pricing model, according to which the price of the commodity is assumed to follow a *two-state discrete jump* process:



If the price of the commodity is P in period t, then its price in period t+1 will be

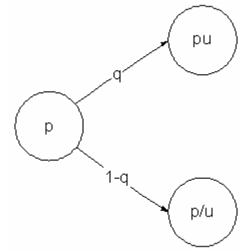
- *Pu* with a certain probability *q*, and
- P/u with probability $1-q$,

where

$$u = e^{\sigma\sqrt{\Delta t}} > 1$$

$$q = \frac{1}{2} + \frac{\sqrt{\Delta t}}{2\sigma}\left(r - \frac{1}{2}\sigma^2\right)$$

$$\delta = e^{-r\Delta t}$$

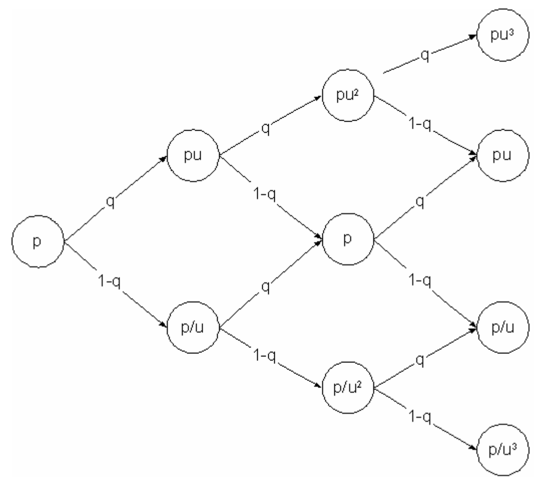


Here,

r is the annual rate of interest (continuously compounded),

σ is the annualized volatility of the commodity price,

Δt is the length of a period in years



If we choose the length of the time periods (Δt) sufficiently short, this gives a reasonably close approximation to reality.

Sample data:

P = 2 = current commodity price

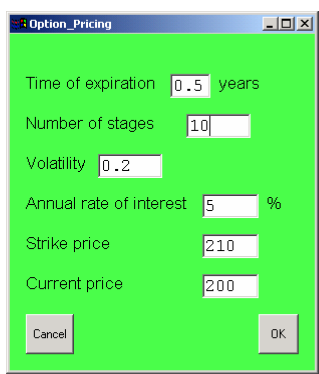
P' = 2.1 = "strike" price

r = 5% annual interest rate

T = 0.5 years (time to expiration of option)

σ = 0.2 annual volatility of commodity price

Δt = .01 year



The 50 *stages* are each of length Δt = 0.01 year.

DP MODEL

The states of the DP model are the possible commodity prices $p \in \left\{p_0 u^i \mid i = -(N+1), -N, \ldots, 0, \ldots, N, (N+1)\right\}$

The two decisions at each stage are $x \in \{\text{KEEP, EXERCISE}\}$

The transition probabilities are:

$$P_{ij}^x = \begin{cases} q & \text{if } j = ui \\ 1-q & \text{if } j = i/u \\ 0 & \text{otherwise} \end{cases}$$

The *reward* function is

$$g(i,x)=\begin{cases}0 & \text{if } x=\text{"keep"}\\ \overline{p}-i & \text{if } x=\text{"exercise option"}\end{cases}$$

The *optimal value* function is the expected value of the option:

$$f_t(i)=\max\begin{cases}\overline{p}-i\\ q\delta f_{t-1}(iu)+(1-q)\delta f_{t-1}\left(i/u\right)\end{cases}$$

with the *post-terminal condition*:

$$f_{N+1}(i)=0$$

APL function

```
     ∇ z←F N;t;v
[1]    ⍝
[2]    ⍝        Option pricing
[3]    ⍝
[4]    :if N=0
[5]      z←((⍴s)⍴0),-BIG
[6]    :else
[7]    ⍝ Recursive def'n of optimal value function
[8]    v←(F N-1)[TRANSITION(⌊/s)⌈(⌈/s)⌊s∘.×(2⍴u)∘.*d]
[9]    v[;2;1]←v[;2;2]←Strike-s
[10]   z←(q,1-q) Maximize_E v
[11]   :endif
[12]
     ∇
```


s \ x:	(hold) 0	(execute) 1	Maximum
127.9	0.000	82.119	82.119
133.7	0.000	76.270	76.270
139.8	0.000	70.153	70.153
146.2	0.000	63.757	63.757
152.9	0.000	57.069	57.069
159.9	0.000	50.074	50.074
167.2	0.000	42.760	42.760
174.9	0.000	35.111	35.111
182.9	0.000	27.112	27.112
191.3	0.000	18.747	18.747
200.0	0.000	10.000	10.000
209.1	0.000	0.853	0.853
218.7	0.000	¯8.713	0.000
228.7	0.000	¯18.716	0.000
239.2	0.000	¯29.177	0.000
250.1	0.000	¯40.116	0.000
261.6	0.000	¯51.555	0.000
273.5	0.000	¯63.518	0.000
286.0	0.000	¯76.028	0.000
299.1	0.000	¯89.109	0.000
312.8	0.000	¯102.790	0.000

Stage #1

As we would expect, at the final stage it is optimal to execute the option if & only if the current price of the commodity is less than the strike price!

s \ x:	(hold) 0	(execute) 1	Maximum
127.9	79.096	82.119	82.119
133.7	75.935	76.270	76.270
139.8	69.804	70.153	70.153
146.2	63.392	63.757	63.757
152.9	56.686	57.069	57.069
159.9	49.674	50.074	50.074
167.2	42.341	42.760	42.760
174.9	34.673	35.111	35.111
182.9	26.655	27.112	27.112
191.3	18.269	18.747	18.747
200.0	9.500	10.000	10.000
209.1	4.832	0.853	4.832
218.7	0.412	¯8.713	0.412
228.7	0.000	¯18.716	0.000
239.2	0.000	¯29.177	0.000
250.1	0.000	¯40.116	0.000
261.6	0.000	¯51.555	0.000
273.5	0.000	¯63.518	0.000
286.0	0.000	¯76.028	0.000
299.1	0.000	¯89.109	0.000
312.8	0.000	¯102.790	0.000

….etc.

Stage #2

In the next-to-last stage, it becomes optimal to hold the option if current price is only slightly below strike price!

s \ x:	***(hold)*** 0	***(execute)*** 1	Maximum
127.9	79.096	82.119	82.119
133.7	75.935	76.270	76.270
139.8	69.804	70.153	70.153
146.2	63.392	63.757	63.757
152.9	56.686	57.069	57.069
159.9	49.674	50.074	50.074
167.2	42.341	42.760	42.760
174.9	34.826	35.111	35.111
182.9	27.439	27.112	27.439
191.3	20.755	18.747	20.755
200.0	14.625	10.000	14.625
209.1	10.145	0.853	10.145
218.7	6.149	¯8.713	6.149
228.7	3.848	¯18.716	3.848
239.2	1.883	¯29.177	1.883
250.1	1.044	¯40.116	1.044
261.6	0.377	¯51.555	0.377
273.5	0.179	¯63.518	0.179
286.0	0.040	¯76.028	0.040
299.1	0.014	¯89.109	0.014
312.8	0.001	¯102.790	0.001

Stage #10 (initial stage)

How big a "killing" must be possible in order to execute the option immediately?

Current State	Optimal Decision	Optimal Value
127.8814638	exec	82.119
133.7303061	exec	76.270
139.8466535	exec	70.153
146.2427409	exec	63.757
152.9313624	exec	57.069
159.9258977	exec	50.074
167.2403381	exec	42.760
174.889315	exec	35.111
182.8881287	keep	27.439
191.2527797	keep	20.755
200	keep	14.625
209.147287	keep	10.145
218.7129382	keep	6.149
228.7160883	keep	3.848
239.1767467	keep	1.883
250.1158384	keep	1.044
261.5552452	keep	0.377
273.5178496	keep	0.179
286.0275809	keep	0.040
299.1094627	keep	0.014
312.7896632	keep	0.001

Stage #10 (initial stage)

⇒value of option is $14.63 (if current price is $200)