Arithmetic-Geometric Mean Inequality

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The Arithmetic-Geometric Mean Inequality

Simplest case: Given two positive numbers a&b,

their arithmetic mean $\frac{1}{2}a + \frac{1}{2}b$

is greater than or equal to their geometric mean $\sqrt{ab}$

i.e., $\boxed{\frac{1}{2}a + \frac{1}{2}b \geq a^{\frac{1}{2}} b^{\frac{1}{2}}}$

with equality if & only if $a = b$

Arithmetic-Geometric Mean Inequality

$$\boxed{\frac{1}{2}a + \frac{1}{2}b \geq a^{\frac{1}{2}} b^{\frac{1}{2}}}$$

For example, let a=2 & b=8. Then this inequality is

$$5 = \underbrace{\frac{1}{2}\times 2 + \frac{1}{2}\times 8}_{\text{Arithmetic mean}} \geq \underbrace{\sqrt{2\times 8} = 4}_{\text{Geometric Mean}}$$

If a=4 & b=9,

$$6.5 = \underbrace{\frac{1}{2}\times 4 + \frac{1}{2}\times 9}_{\text{Arithmetic mean}} \geq \underbrace{\sqrt{4\times 9} = 6}_{\text{Geometric Mean}}$$

Arithmetic-Geometric Mean Inequality

$$\boxed{\frac{1}{2}a + \frac{1}{2}b \geq a^{\frac{1}{2}} b^{\frac{1}{2}}}$$

Proof: Let α & β be real numbers

and $a = \alpha^2 \geq 0$

$b = \beta^2 \geq 0$

Then $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 \geq 0$

$\Longrightarrow \alpha^2 + \beta^2 \geq 2\alpha\beta$

$\Longrightarrow \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 \geq \alpha\beta \Longrightarrow \frac{1}{2}a + \frac{1}{2}b \geq \sqrt{ab}$

The Arithmetic-Geometric Mean Inequality

The General Case: Let $x_1, x_2, \ldots x_n > 0$

and $\delta_1, \delta_2, \ldots \delta_n \geq 0$ and $\sum_{i=1}^{n} \delta_i = 1$

Then $\boxed{\sum_{i=1}^{n} \delta_i x_i \geq \prod_{i=1}^{n} x_i^{\delta_i}}$

with equality if & only if $x_1 = x_2 = \ldots = x_n$

The Arithmetic-Geometric Mean Inequality

$$\boxed{\sum_{i=1}^{n} \delta_i x_i \geq \prod_{i=1}^{n} x_i^{\delta_i}}$$

If we let n=2, and $\delta_i = \frac{1}{2}$, then we obtain the earlier inequality,

$$\boxed{\frac{1}{2}a + \frac{1}{2}b \geq a^{\frac{1}{2}} b^{\frac{1}{2}}}$$

For the proof of the Arithmetic-Geometric Mean Inequality, we need the fact that $f(x) = -\ln x$ is a strictly convex function of x in its domain, namely x such that $x > 0$:

$f(x) = -\ln x$

$\Longrightarrow f'(x) = -x^{-1}$

$\Longrightarrow f''(x) = x^{-2}$

$\Longrightarrow f''(x) > 0$ for $x > 0$

$\Longrightarrow$ f is strictly convex

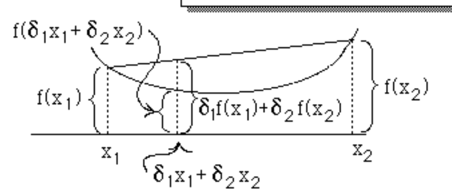
A convex function f has the property that

for any x_1 & x_2 in its domain,

and

$\delta_1 > 0$ & $\delta_2 > 0$ such that $\delta_1 + \delta_2 = 1$,

$$\boxed{\delta_1 f(x_1) + \delta_2 f(x_2) \geq f(\delta_1 x_1 + \delta_2 x_2)}$$



If f is ***strictly*** convex, then there is equality if & only if $x_1 = x_2$

More generally, for a convex function f,

if $x_1, x_2, \ldots x_n$ are in its domain,

and we are given a "weight" δ_i for each x_i such that

$$\sum_{i=1}^{n} \delta_i = \mathbf{1}, \quad \delta_i \geq 0$$

then

$$\underbrace{\delta_1 f(x_1) + \delta_2 f(x_2) + \ldots + \delta_n f(x_n)}_{\text{convex combination of function values}} \geq \underbrace{f(\delta_1 x_1 + \delta_2 x + \ldots + \delta_n x_n)}_{\text{function evaluated at the convex combination of the } x_i\text{'s}}$$

If f is strictly convex, then there is equality above if & only if

$$x_1 = x_2 = \ldots = x_n$$

Arithmetic-Geometric Mean Inequality:

Proof: For $x > 0$, the function $f(x) = -\ln x$ is strictly convex

Consequently, if $x_1, x_2, \ldots x_n > 0$ and $\sum_{i=1}^{n} \delta_i = \mathbf{1}$, $\delta_i \geq 0$

then $\delta_1 f(x_1) + \delta_2 f(x_2) + \ldots + \delta_n f(x_n) \geq f(\delta_1 x_1 + \delta_2 x + \ldots + \delta_n x_n)$

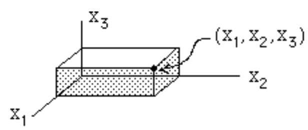
i.e., $-\delta_1 \ln x_1 - \delta_2 \ln x_2 - \ldots - \delta_n \ln x_n \geq -\ln(\delta_1 x_1 + \delta_2 x + \ldots + \delta_n x_n)$

$$\delta_1 \ln x_1 + \delta_2 \ln x_2 + \ldots + \delta_n \ln x_n \leq \ln(\delta_1 x_1 + \delta_2 x + \ldots + \delta_n x_n)$$

$$\ln \prod_{i=1}^{n} x_i^{\delta_i} \leq \ln\left(\sum_{i=1}^{n} \delta_i x_i\right)$$

Since the log function is strictly increasing, $\prod_{i=1}^{n} x_i^{\delta_i} \leq \sum_{i=1}^{n} \delta_i x_i$

$$\boxed{\sum_{i=1}^{n} \delta_i x_i \geq \prod_{i=1}^{n} x_i^{\delta_i}}$$



Example: Find the dimensions of the open rectangular box with a fixed surface area S_0 having the greatest volume.

Let the dimensions be denoted by $x_1, x_2, \& x_3$
Then the volume is $V(x) = x_1 x_2 x_3$,
and the surface area is

$$So = \underbrace{x_1 x_2}_{\text{area of bottom}} + \underbrace{2 x_2 x_3}_{\text{area of front \& back}} + \underbrace{2 x_1 x_3}_{\text{area of two ends}}$$

$$S_0 = x_1 x_2 + 2 x_2 x_3 + 2 x_1 x_3 = 3\left[\frac{x_1 x_2 + 2 x_2 x_3 + 2 x_1 x_3}{3}\right]$$

$$S_0 = 3\left[\tfrac{1}{3}(x_1 x_2) + \tfrac{1}{3}(2 x_2 x_3) + \tfrac{1}{3}(2 x_1 x_3)\right]$$

$$\underset{\text{by the A-G inequality}}{\geq} 3 \left(x_1 x_2\right)^{1/3} \left(2 x_2 x_3\right)^{1/3} \left(2 x_1 x_3\right)^{1/3}$$

$$= 3 \cdot 4^{1/3} \left(x_1^2 x_2^2 x_3^2\right)^{1/3} = 3 \cdot 4^{1/3} [V(x)]^{2/3}$$

By the A-G Mean Inequality, then,

$$3 \cdot 4^{1/3} [V(x)]^{2/3} \leq S_0 \quad \text{for all } x,$$

with equality if & only if the three terms

$$(x_1 x_2),\ (2 x_2 x_3),\ \text{and}\ (2 x_1 x_3)$$

are equal.

$$S_0 = x_1 x_2 + 2 x_2 x_3 + 2 x_1 x_3$$

$$\Longrightarrow \quad x_1 x_2 = 2 x_2 x_3 = 2 x_1 x_3 = \tfrac{1}{3} S_0$$

We can solve for $x_1, x_2, \& x_3$ by using logarithms

$$x_1 x_2 = 2 x_2 x_3 = 2 x_1 x_3 = \tfrac{1}{3} S_0$$

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$$\begin{cases} x_1 x_2 = \tfrac{1}{3} S_0 \Longrightarrow \\ 2 x_2 x_3 = \tfrac{1}{3} S_0 \Longrightarrow \\ 2 x_1 x_3 = \tfrac{1}{3} S_0 \Longrightarrow \end{cases} \begin{cases} \ln x_1 + \ln x_2 = \ln(\tfrac{1}{3} S_0) \\ \ln 2 + \ln x_2 + \ln x_3 = \ln(\tfrac{1}{3} S_0) \\ \ln 2 + \ln x_1 + \ln x_3 = \ln(\tfrac{1}{3} S_0) \end{cases}$$

This is a linear system of equations in $(\ln x_i)$,

Let $z_i = \ln x_i$ and $K = \ln(\tfrac{1}{3} S_0)$
Then

$$\begin{cases} \ln x_1 + \ln x_2 = \ln(\tfrac{1}{3} S_0) \\ \ln 2 + \ln x_2 + \ln x_3 = \ln(\tfrac{1}{3} S_0) \\ \ln 2 + \ln x_1 + \ln x_3 = \ln(\tfrac{1}{3} S_0) \end{cases} \Rightarrow \begin{cases} z_1 + z_3 = K \\ \ln 2 + z_2 + z_3 = K \\ \ln 2 + z_1 + z_2 = K \end{cases}$$

$$\Rightarrow \begin{cases} z_1 + z_2 = K \\ z_2 + z_3 = K - \ln 2 \\ z_3 = \frac{K}{2} - \ln 2 \end{cases} \Rightarrow \begin{cases} z_1 = \frac{K}{2} \\ z_2 = \frac{K}{2} \\ z_3 = \frac{K}{2} - \ln 2 \end{cases}$$

$$\begin{cases} \ln x_1 = z_1 = \frac{K}{2} \\ \ln x_2 = z_2 = \frac{K}{2} \\ \ln x_3 = z_3 = \frac{K}{2} - \ln 2 \\ K \equiv \ln(\tfrac{1}{3} S_0) \end{cases}$$

which yields the solution

$$x_1 = x_2 = \sqrt{\tfrac{1}{3} S_0}$$

$$x_3 = \tfrac{1}{2} \sqrt{\tfrac{1}{3} S_0}$$

so that the volume of the box is

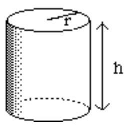
$$V(x) = x_1 x_2 x_3 = \tfrac{1}{2} \left(\frac{S_0}{3}\right)^{3/2}$$

$\boxed{\text{Example}}$

Maximize the volume of a cylindrical can given a total cost C_0 if

the cost of the top & bottom of the can is C_1 cents/square inch

the cost of the side of the can is C_2 cents/square inch



Volume is $V(r,h) = \pi r^2 h$

Cost is $\underbrace{2\pi r^2 C_1}_{\text{cost of top \& bottom}} + \underbrace{2\pi r h C_2}_{\text{cost of side}} = C_0$

$$C_0 = 2\pi r^2 C_1 + 2\pi r h C_2 = 4\left(\frac{\pi r^2 C_1}{2} + \frac{\pi r h C_2}{2}\right)$$

$$\frac{\pi r^2 C_1}{2} + \frac{\pi r h C_2}{2} \geq \left(\pi r^2 C_1\right)^{1/2}\left(\pi r h C_2\right)^{1/2} = \pi r^{3/2} h^{1/2} \left(C_1 C_2\right)^{1/2}$$

by the A-G Mean Inequality

Not a constant multiple of the volume: $V(r,h) = \pi r^2 h$

Unfortunately, we cannot proceed as before. We need to "split" C_0 into a sum of terms differently.

$$\boxed{C_0 = 2\pi r^2 C_1 + 2\pi r h C_2}$$

"Split" the total cost into three terms

$$C_0 = 2\pi r^2 C_1 + \pi r h C_2 + \pi r h C_2$$

$$= 3\left(\tfrac{1}{3}\left(2\pi r^2 C_1\right) + \tfrac{1}{3}\left(\pi r h C_2\right) + \tfrac{1}{3}\left(\pi r h C_2\right)\right)$$

We now again apply the Arithmetic-Geometric Mean Inequality to the sum within the braces, with weights equal to $\frac{1}{3}$ for each term.

$$C_0 = 3\left(\tfrac{1}{3}\left(2\pi r^2 C_1\right) + \tfrac{1}{3}\left(\pi r h C_2\right) + \tfrac{1}{3}\left(\pi r h C_2\right)\right)$$

$$\geq 3\left(2\pi r^2 C_1\right)^{1/3}\left(\pi r h C_2\right)^{1/3}\left(\pi r h C_2\right)^{1/3} = 3 C_1^{1/3} C_2^{2/3} 2^{1/3} \underbrace{\pi r^{4/3} h^{2/3}}_{\pi^{1/3}\left(\pi r^2 h\right)^{2/3}}$$

$$C_0 \geq 3 C_1^{1/3} C_2^{2/3} 2^{1/3} \pi^{1/3} V(r,h)^{2/3}$$

by the A-G Mean Inequality

with equality if & only if the three terms are equal!

That is, by the A-G Mean Inequality,

$$C_0 \geq 3 C_1^{1/3} C_2^{2/3} 2^{1/3} \pi^{1/3} V(r,h)^{2/3} \quad \text{for all } r \,\&\, h$$

with equality if & only if the terms are equal:

$$\left(2\pi r^2 C_1\right) = \left(\pi r h C_2\right) = \left(\pi r h C_2\right) = \tfrac{1}{3} C_0$$

$$\Rightarrow r = \sqrt{\frac{\frac{1}{3} C_0}{2\pi C_1}} \Rightarrow h = \frac{\frac{1}{3} C_0}{\pi r C_2} = \frac{1}{C_2}\sqrt{\frac{2 C_0 C_1}{3\pi}}$$

The "trick" is knowing how to "split" the total cost into several terms.

Geometric Programming

"Geometric Programming" is an optimization technique which determines what fraction of the cost is to be attributed to each term, i.e., determines the "weights" in the Arithmetic-Geometric Mean Inequality.

$\boxed{\text{Example}}$ Minimize $f(x) = C_1 x^3 + \frac{C_2}{x}$

$$f(x) = C_1 x^3 + \frac{1}{3}\frac{C_2}{x} + \frac{1}{3}\frac{C_2}{x} + \frac{1}{3}\frac{C_2}{x}$$

$$= 4\left(\frac{1}{4}\left(C_1 x^3\right) + \frac{1}{4}\left(\frac{1}{3}\frac{C_2}{x}\right) + \frac{1}{4}\left(\frac{1}{3}\frac{C_2}{x}\right) + \frac{1}{4}\left(\frac{1}{3}\frac{C_2}{x}\right)\right)$$

$$\geq 4\left(\left(C_1 x^3\right)^{1/4}\left(\frac{1}{3}\frac{C_2}{x}\right)^{1/4}\left(\frac{1}{3}\frac{C_2}{x}\right)^{1/4}\left(\frac{1}{3}\frac{C_2}{x}\right)^{1/4}\right) = 4\underbrace{\left(\tfrac{1}{3}\right)^{3/4} C_1^{1/4} C_2^{3/4}}_{\text{doesn't depend on } x!}$$

by A-G Mean Inequality

$$\text{Minimize } f(x) = C_1 x^3 + \frac{C_2}{x} \geq 4\left(\tfrac{1}{3}\right)^{3/4} C_1^{1/4} C_2^{3/4}$$

with equality if & only if

$$C_1 x^3 = \frac{1}{3}\frac{C_2}{x} = \frac{1}{3}\frac{C_2}{x} = \frac{1}{3}\frac{C_2}{x} = \left(\tfrac{1}{3}\right)^{3/4} C_1^{1/4} C_2^{3/4}$$

$$\Rightarrow x = \left(\frac{1}{3}\right)^{1/4} C_1^{-1/4} C_2^{1/4}$$

Again, the "trick" is knowing how to "split" the objective, i.e., allocating a fourth of the total cost to the first term, and three-fourths to the second term.