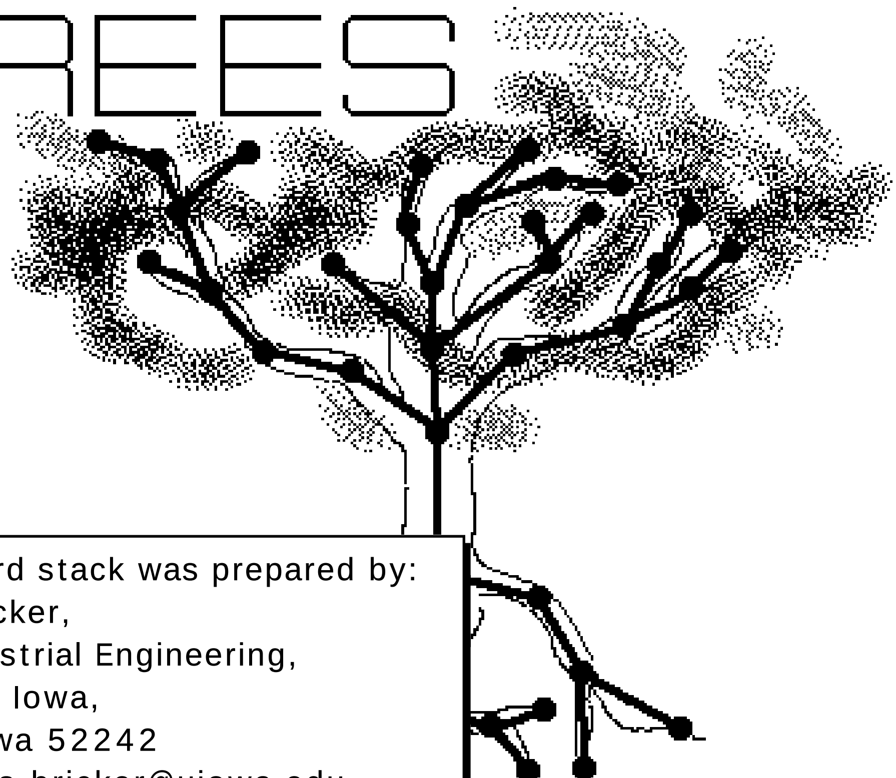
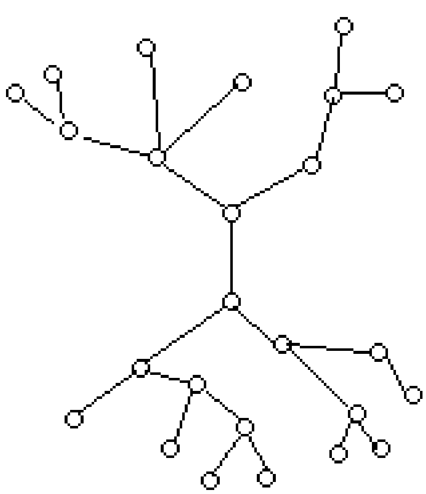
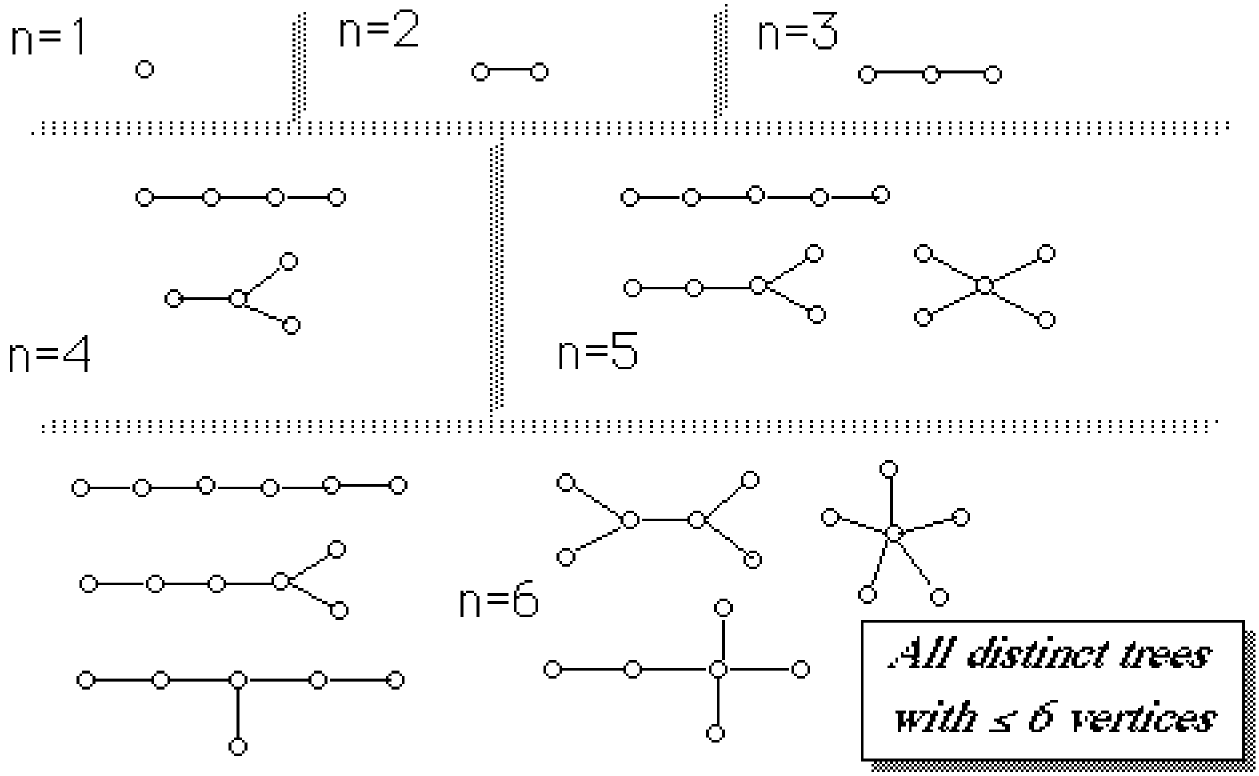
TREES



This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dennis-bricker@uiowa.edu

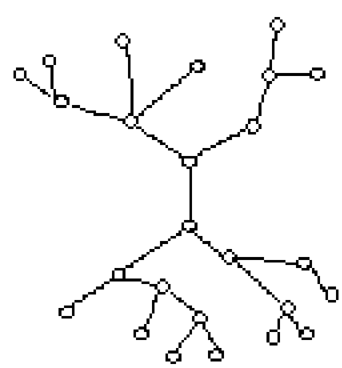
Tree : a connected graph without cycles



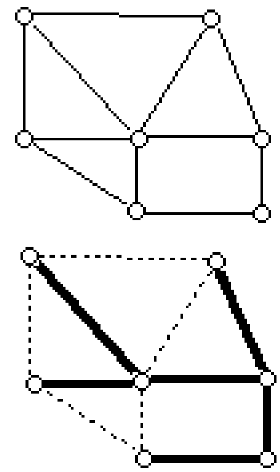


The following statements about a graph G are equivalent:

- G is a tree
- G is connected with *n* vertices and *n-1* edges
- G has *n* vertices, *n-1* edges, and no cycles
- G is such that each pair of vertices is connected by a *unique* elementary chain



Spanning tree



A spanning tree of a connected graph G=(V,A) is a tree with vertex set V and an edge set which is a subset of A

Minimum spanning tree

A minimum spanning tree of a *network* is a spanning tree the sum of whose edge lengths are minimal.

Two algorithms for MST problem:

☞ Prim's Algorithm

Beginning with a single node, at each iteration a tree is obtained by adding an edge & node, until ALL nodes have been included.

☞ Kruskal's Algorithm

Beginning with N trees, each consisting of a single node, at each iteration two trees are combined by adding an edge, until a single tree is obtained.

Finding a Minimum Spanning Tree (MST) of a Network (Prim's algorithm)

Step 1 (Setup)

Select any node to begin the tree

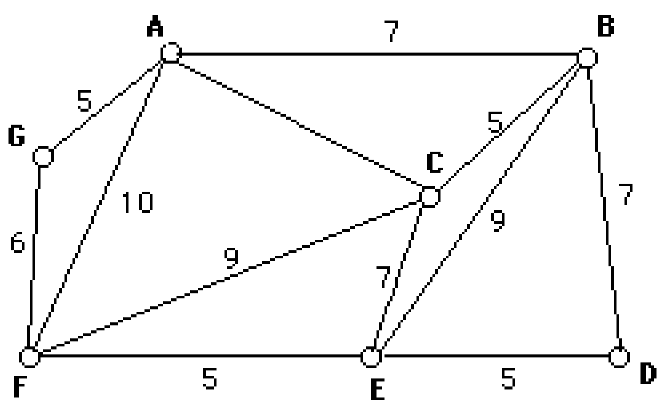
Step 2 (Addition)

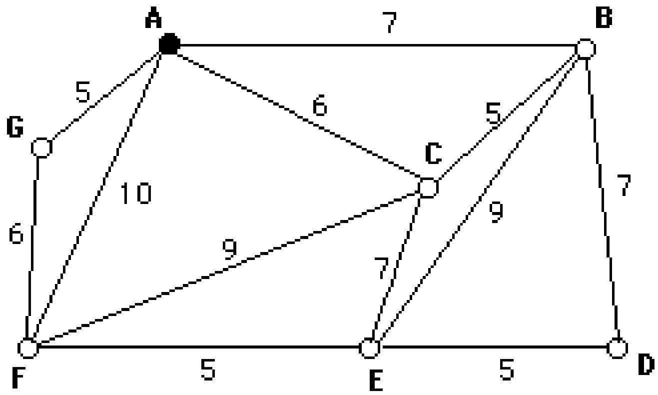
Find a node NOT currently in the tree which is nearest to the set of nodes IN the tree. Add that node and the connecting edge to the tree

Step 3 (Stopping criterion)

If all nodes are in the tree, STOP; otherwise return to step 2

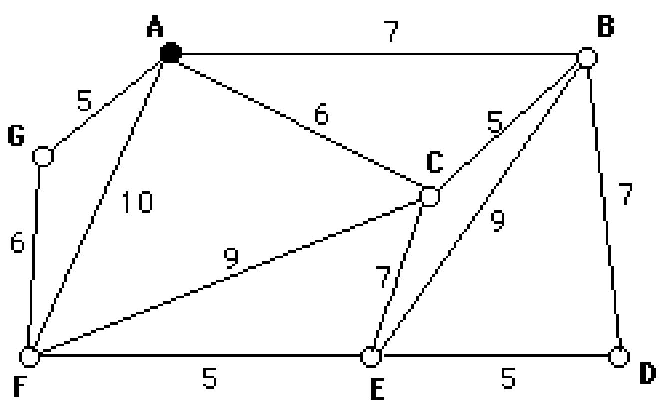
Example: Prim's algorithm for MST





Initially, the tree is empty.

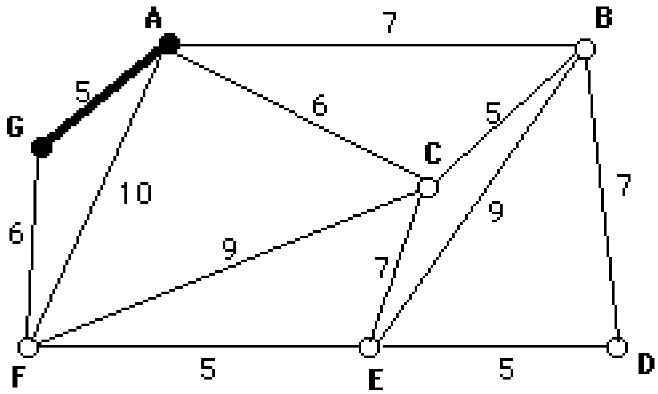
Select (arbitrarily) node A to add to the tree.



Find the node which is nearest to the nodes of the tree (i.e., node A)

This is node G.

Add it (and edge [A,G]) to the tree

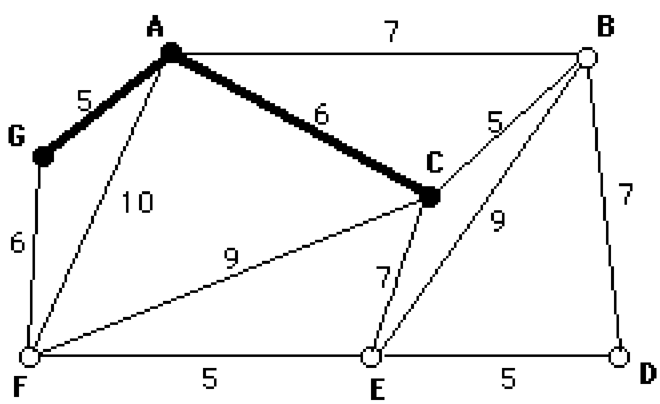


Find the node in the set {B,C,D,E,F} (not in the tree) which is nearest to the nodes {A,G} which are in the tree.

In this case there is a tie!

Break the tie arbitrarily, by selecting node C.

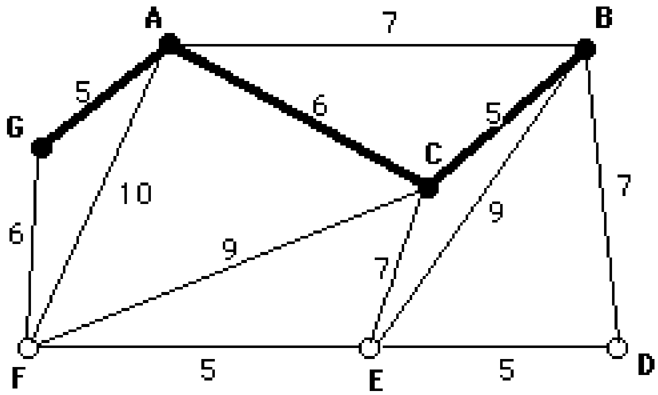
Add node C (and edge [A,C]) to the tree



Find the node from the set {B,D,E,F} (not in the tree) which is nearest to the nodes {A,C,G} (in the tree)

This is node B, a distance 5 from the tree.

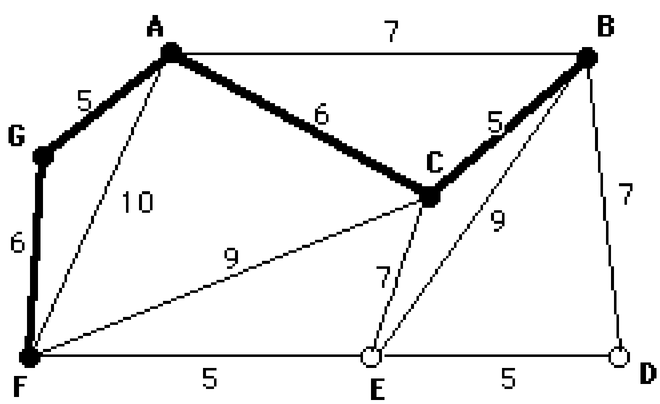
Add the node B (and the edge [B,C]) to the tree



Find the node from the set {D,E,F} which is nearest to the set of nodes in the tree, {A,B,C,G}.

This is node F, a distance of 6 from the tree.

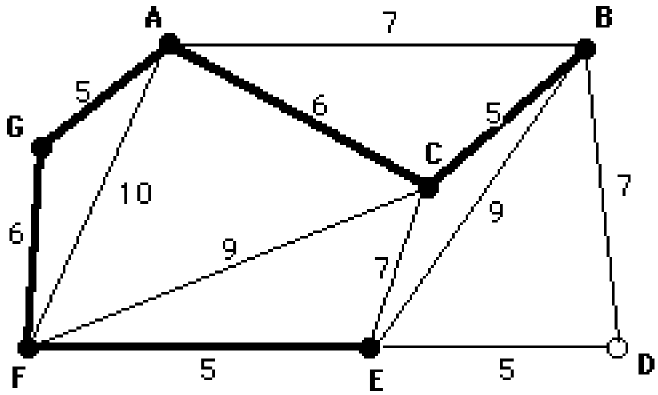
Add node F (and edge [F,G]) to the tree



Find the node from the set {D,E} which is nearest to the nodes in the tree, {A,B,C,F,G}.

This is node E, a distance of 5 from the tree.

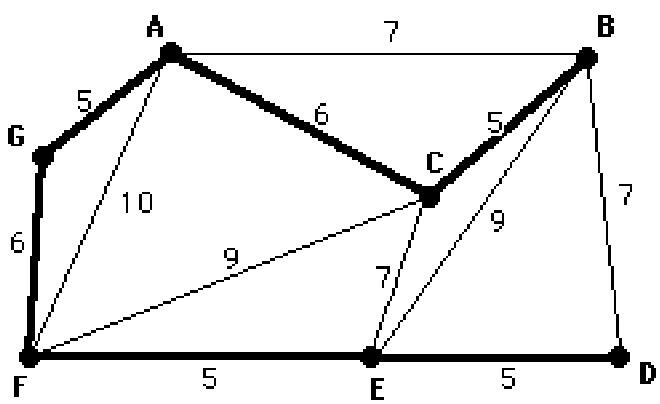
Add node E (and edge [E,F]) to the tree.



Find the node from the set {D} which is nearest to the nodes {A,B,C,E,F,G} in the tree.

This is node D, a distance of 5 from the tree.

Add node D (and edge [D,E]) to the tree.



All nodes are now in the tree, so we stop!

Example: **Alaska Gas Transmission Company** is planning to construct a pipeline to supply gas from Alaska's north slope ("NS") to eight U.S. gas companies, denoted by A through H.

Each mile of "right-of-way which is purchased costs an average of $1000.

How should the pipeline be routed to minimize the total cost of the right-of-way?

distances (x100 miles)

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Arbitrarily select a node to begin the tree.

Let's choose node NS.

non-TREE

TREE

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Find the minimum distance from a node NOT in the tree to the node IN the tree.

This is node A.

Add node A (and edge [NS,A]) to the tree.

non-TREE

TREE

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Find the node NOT in the tree which is nearest to the nodes IN the tree.

This is node B, a distance of 12 from node A.

Add node B (& edge [A,B]) to the tree.

non-TREE

TREE: NS, A, B

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Find the node NOT in the tree which is nearest to the nodes IN the tree.

This is node E, a distance of 11 from node B.

Add node E (& edge [B,E]) to the tree.

non-TREE

TREE: NS, A, B, E

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Find the node NOT in the tree which is nearest to the nodes IN the tree.

This is node D, which is a distance 5 from node E.

Add node D (& edge [D,E]) to the tree.

non-TREE (columns C, F, G, H) · TREE (rows NS, A, B, D, E)

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Find the node NOT in the tree which is nearest to the nodes IN the tree.

This is node C, a distance of 10 from node D.

Add node C (& edge [C,D]) to the tree.

non-TREE (columns F, G, H) · TREE (rows NS, A, B, C, D, E)

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Find the node NOT in the tree which is nearest to the nodes IN the tree.

This is node H, a distance of 12 from node E.

Add node H (& edge [E,H]) to the tree.

non-TREE

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

TREE

Find the node NOT in the tree which is nearest to the nodes IN the tree.

This is node G, a distance of 8 from node H.

Add node G (& edge [G,H]) to the tree.

non-TREE

TREE

	NS	A	B	C	D	E	F	G	H
NS	0	32	43	41	44	45	53	56	61
A	32	0	12	15	16	17	31	25	32
B	43	12	0	18	12	11	32	26	28
C	41	15	18	0	10	14	23	15	18
D	44	16	12	10	0	5	22	13	16
E	45	17	11	14	5	0	23	15	12
F	53	31	32	23	22	23	0	7	14
G	56	25	26	15	13	15	7	0	8
H	61	32	28	18	16	12	14	8	0

Find the node NOT in the tree which is nearest to the nodes IN the tree.

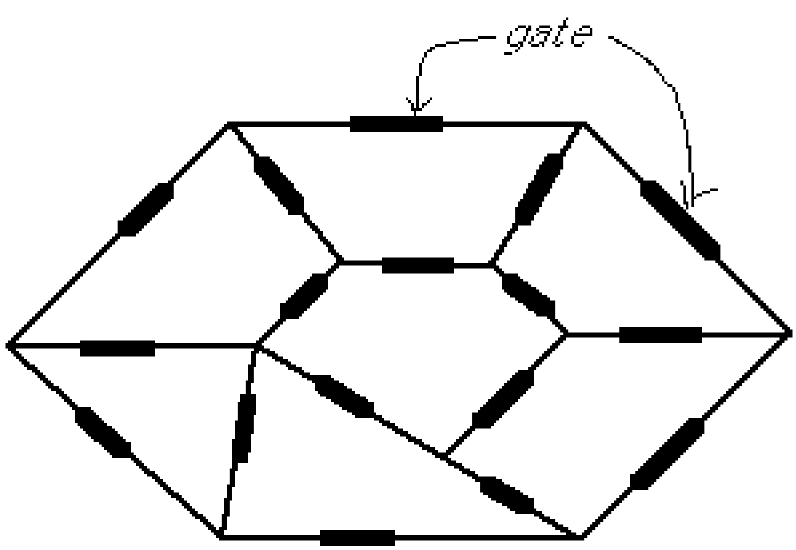
This is node F, a distance of 7 from node G.

Add node F (& edge [F,G]) to the tree.

The tree now spans all nine nodes, and is the Minimum Spanning Tree.

APL code for Prim's MST algorithm

```
    ∇TREE←MST C;IN;OUT;K;L;ROWMIN;MIN;J
     ⍝
     ⍝  Compute Minimum Spanning Tree of a graph
     ⍝
     IN←,1                          ⍝ List of nodes in tree
     OUT←1+⍳¯1+1↑⍴C                 ⍝ List of nodes not yet in
     TREE←(⍴C)⍴0
     LENGTH←0
     ⍝         Find shortest arc joining IN & OUT nodes
     NEXT:ROWMIN←⌊/C[IN;OUT]
     MIN←⌊/ROWMIN
     J←ROWMIN⍳MIN
     ⍝         Add arc from IN node (K) to OUT node (L)
     K←IN[J]
     L←OUT[C[K;OUT]⍳MIN]
     TREE[K;L]←1
     OUT←(L≠OUT)/OUT
     IN←IN,L
     LENGTH←LENGTH+MIN
     →NEXT IF (⍴IN)<1↑⍴C
    ∇
```

JAILBREAK!

- Prisoners have been divided into seven groups by walls
- An outside accomplice plans to help them to escape by blowing up some of the gates, using explosives

HOW CAN HE DO THIS, DESTROYING AS FEW GATES AS POSSIBLE?

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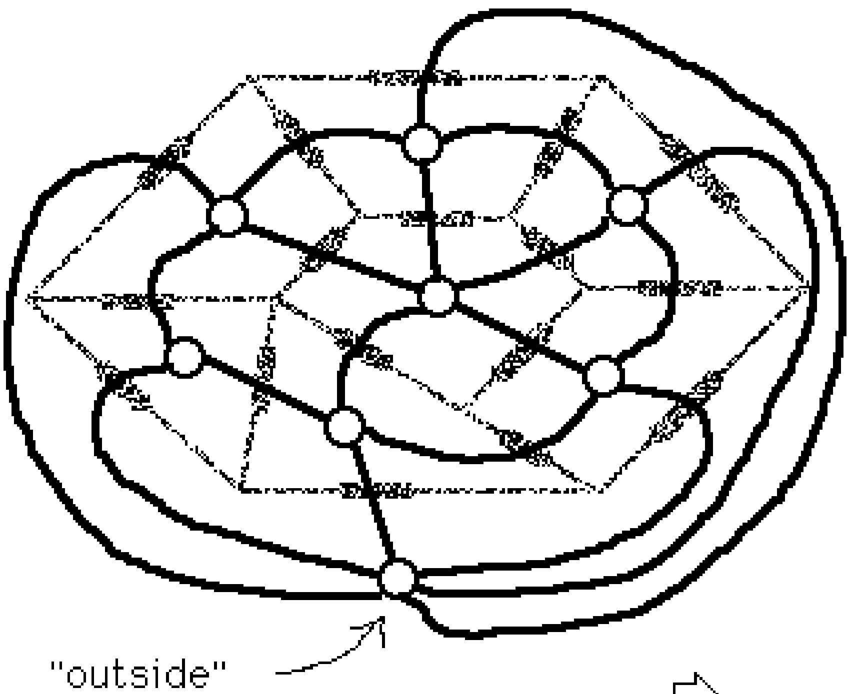
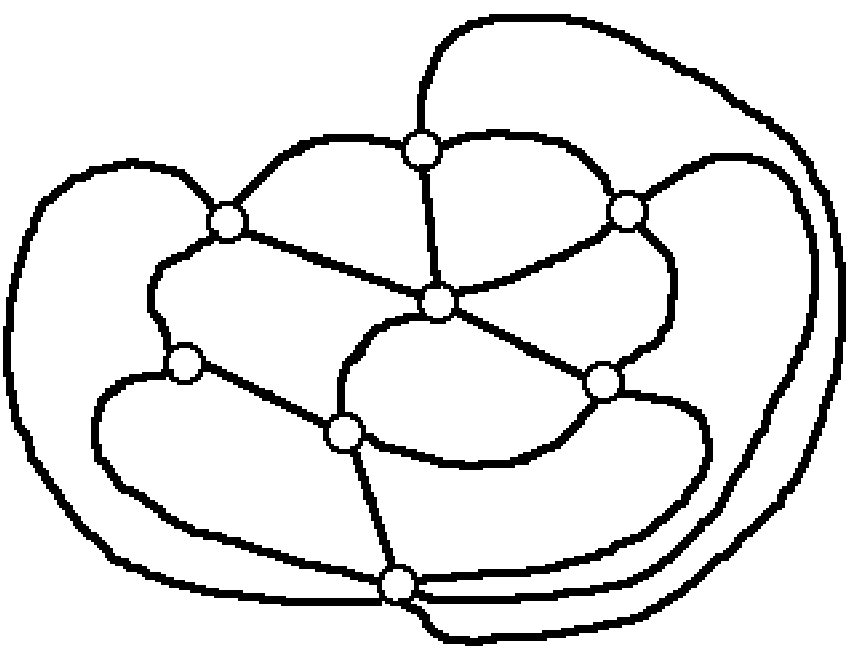


Represent each room, together with the "outside world", by a node, and each gate by an edge.

The problem is to find a spanning tree with the fewest edges!



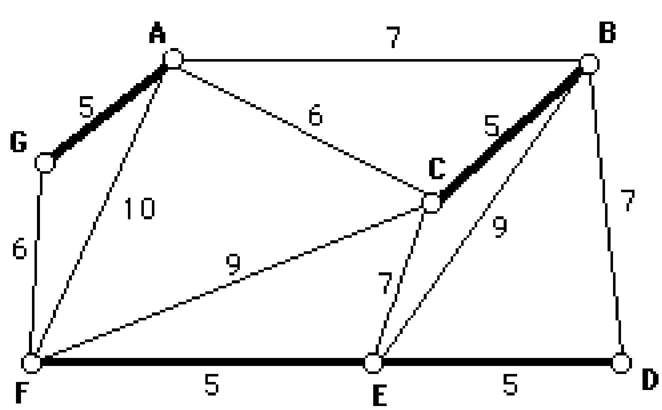
The number of nodes is 8.

All spanning trees will have seven edges!

Kruskal's Algorithm for MST

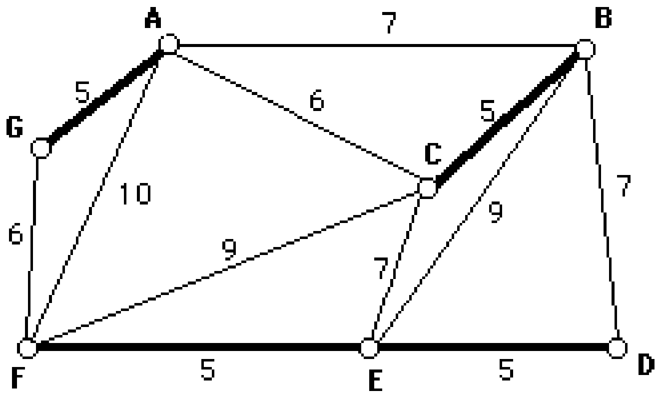
Step 1: Setup
  Let $G_0 = (V, \varnothing)$ and $i=0$
Step 2: Addition of Edge
  Find (x,y) which minimizes $w(x,y)$, and set $w(x,y)= +\infty$
Step 3: Test for cycle
  If the addition of edge (x,y) to the graph G_i would form a cycle, then go to step 2;
  Otherwise, add edge (x,y) to graph G_i and increment i.
Step 4: Test for termination
  If $i < n-1$, then return to step 2.
  Otherwise, stop with G_{n-1} = MST

Example (Kruskal's MST Algorithm)



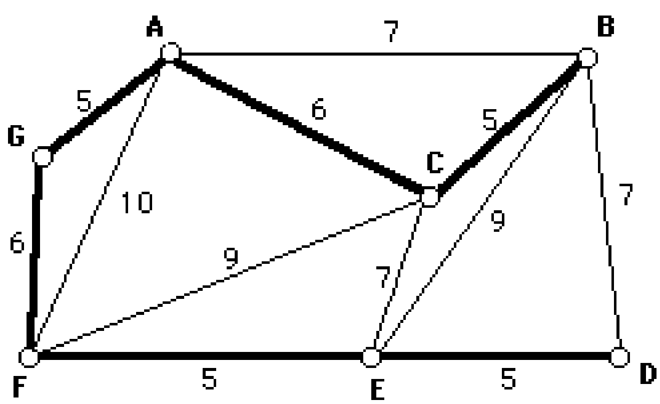
i	Edges in G_i
0	none
1	AG
2	AG, BC
3	AG, BC, DE
4	AG, BC, DE, EF

In each of the first 4 iterations, there is a tie for the minimum-length edge to be added



i	Edges in G_i
0	none
1	AG
2	AG, BC
3	AG, BC, DE
4	AG, BC, DE, EF

Next, there is a tie between edges FG and AC

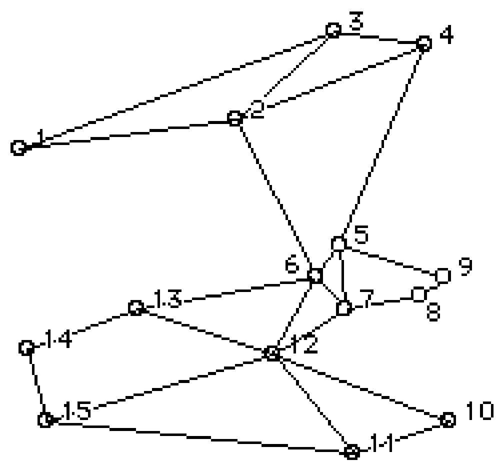


i	Edges in G_i
0	none
1	AG
2	AG, BC
3	AG, BC, DE
4	AG, BC, DE, EF
5	AG, BC, DE, EF, FG
6	AG, BC, DE, EF, FG, AC

Since i=6 = n-1, we terminate.

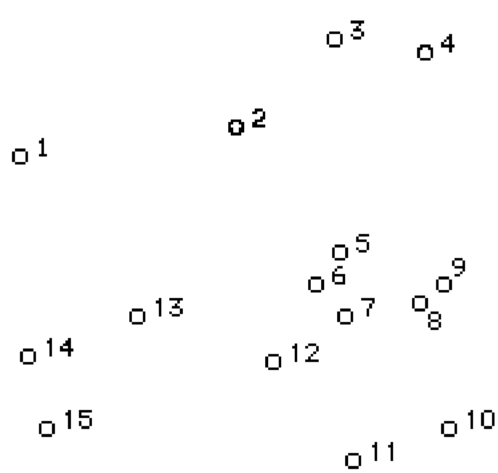
Example (Kruskal's MST Algorithm)

A network with 15 nodes:



i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

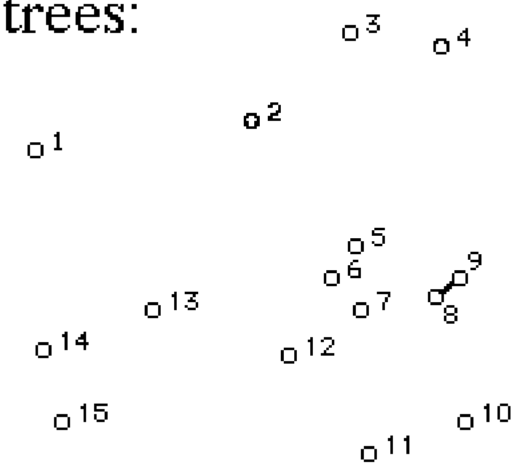
We begin with 15 "trees", each consisting of a single node:



edges sorted according to length

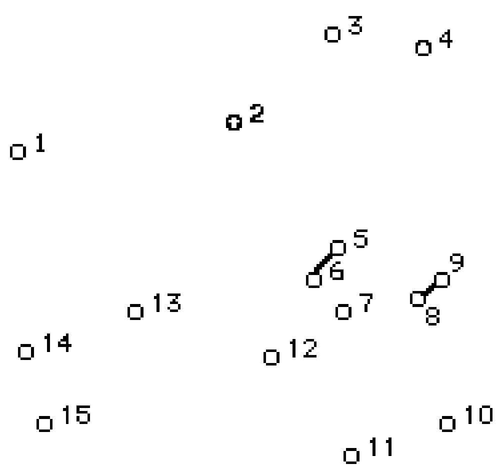
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

The two trees consisting of nodes 8 & 9 are joined, so that we now have 14 trees:



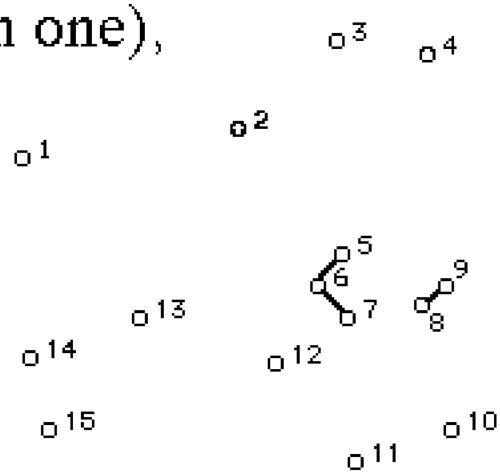
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Next, edge (5,6) is added, which joins two trees, resulting in only 13 trees:



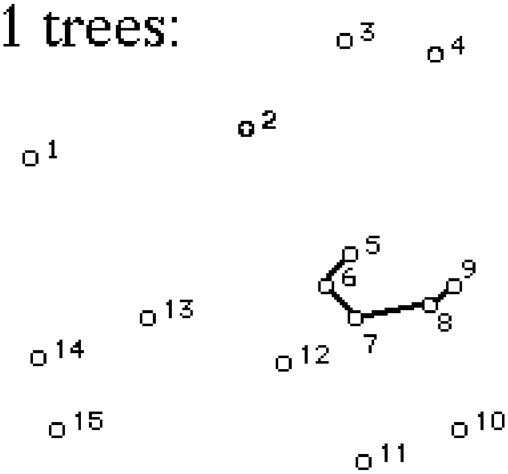
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Edge (6,7) is next added, combining two trees (one with 2 nodes, the other with one), giving us 12 trees:



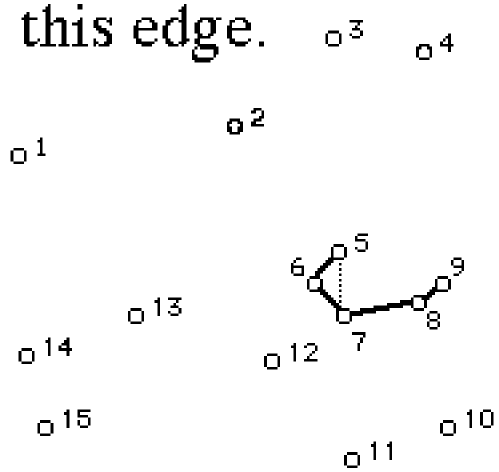
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Edge (7,8) is added, combining trees {5,6,7} and {8,9}, giving us only 11 trees:



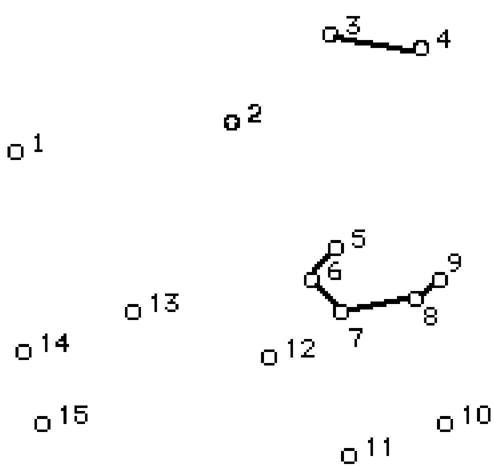
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

If edge (5,7) were added, a cycle 5-7-6-5 would be formed, and so we "skip" this edge.



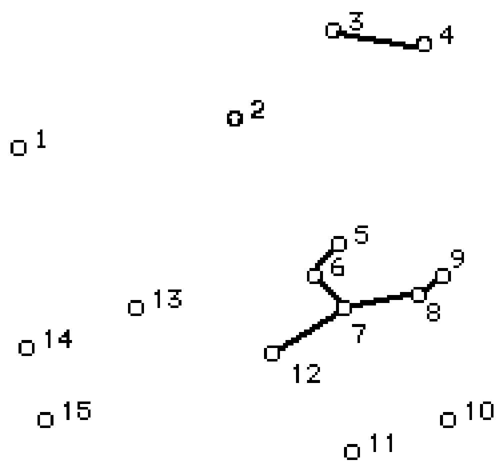
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Edge (3,4) is added next, reducing the number of trees to only 10:



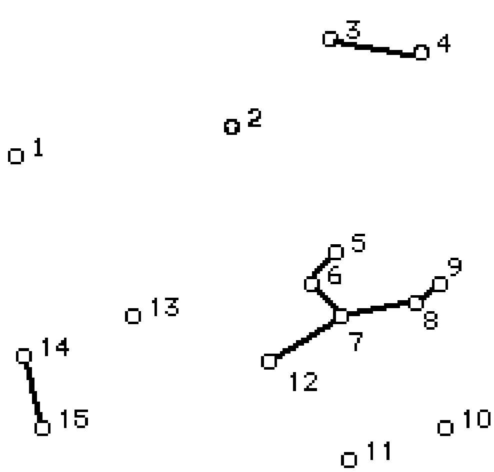
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Edge (7,12) is added, reducing the number of trees to nine:



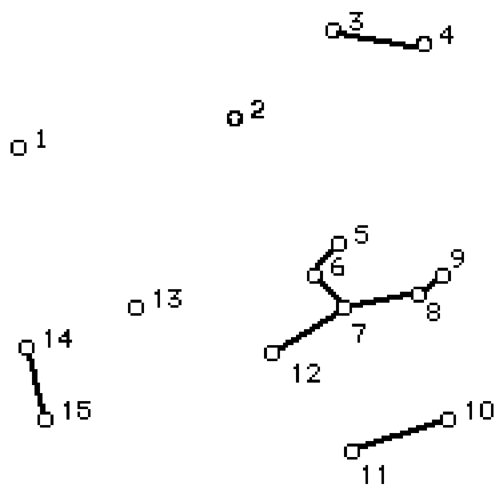
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Edge (14,15) is added, reducing the number of trees to eight:



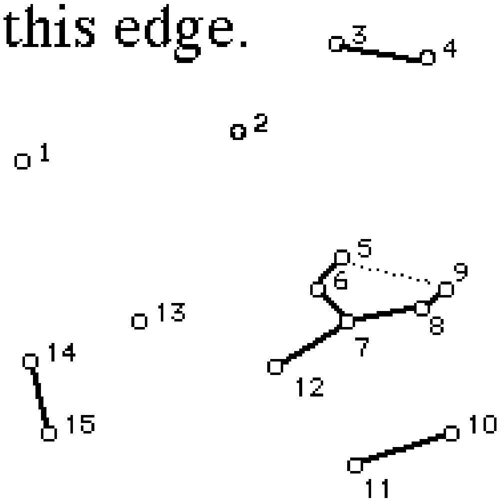
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Adding edge (10,11) reduces the number of trees to seven:



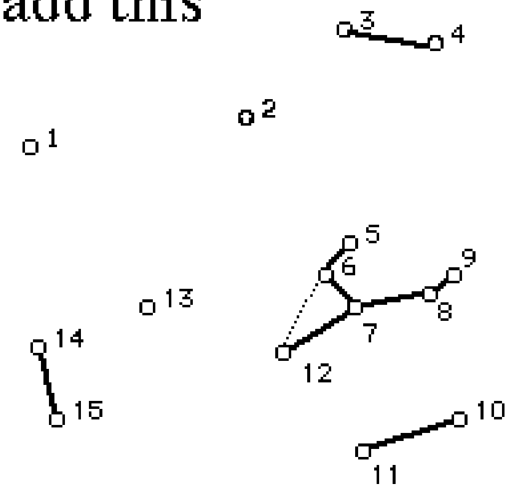
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Adding edge (5,9) would create a cycle (5-9-8-7-6-5) and so we don't add this edge.



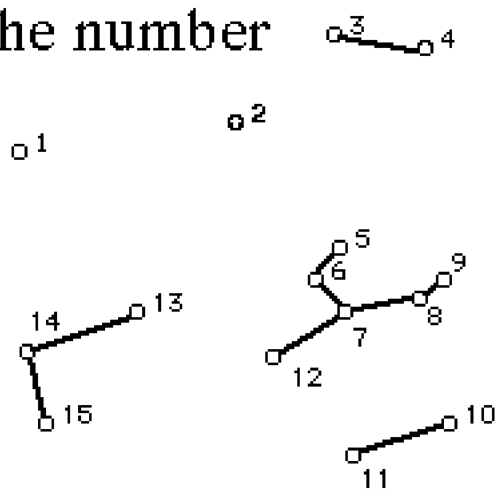
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Adding edge (6,12) also would create a cycle (6-12-7-6), and so we don't add this edge:



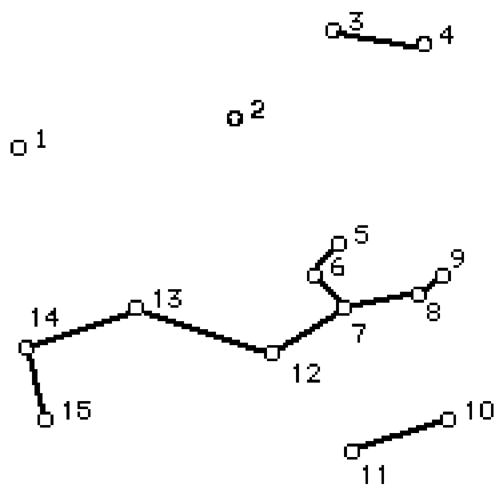
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Adding edge (13,14) doesn't create a cycle, and so we add this edge, reducing the number of trees to only six:



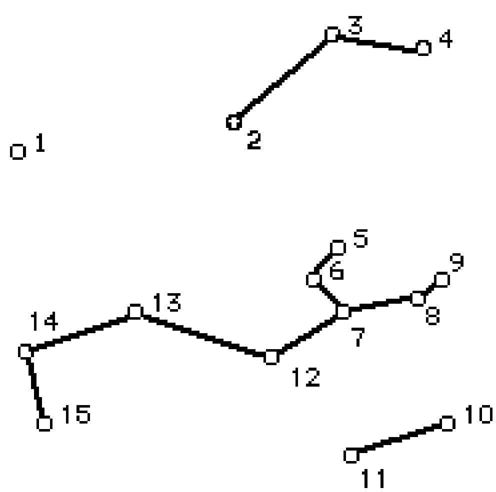
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Edge (12,13) is added, reducing the number of trees to five:



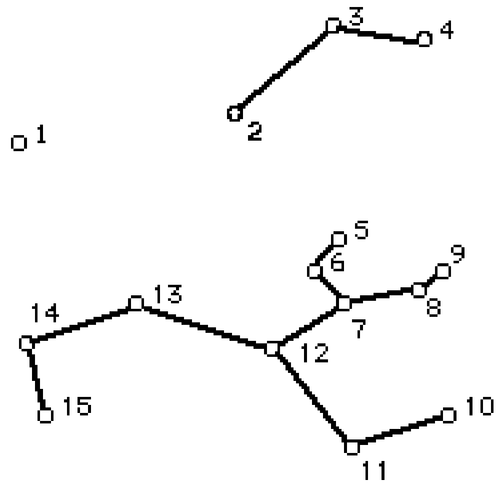
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Edge (2,3) is added next, reducing the number of trees to four:



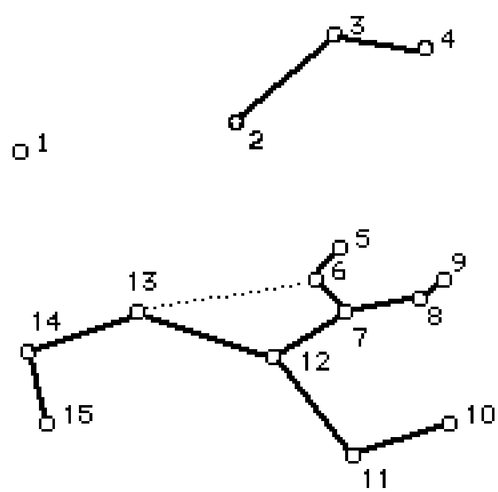
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Next we add edge (11,12), to obtain three trees:



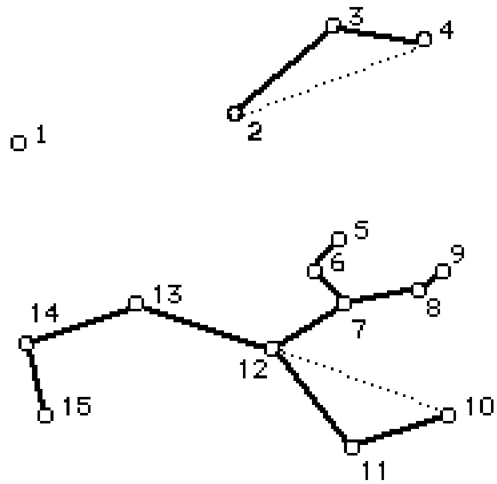
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Adding edge (6,13) would form a cycle, so we skip it:



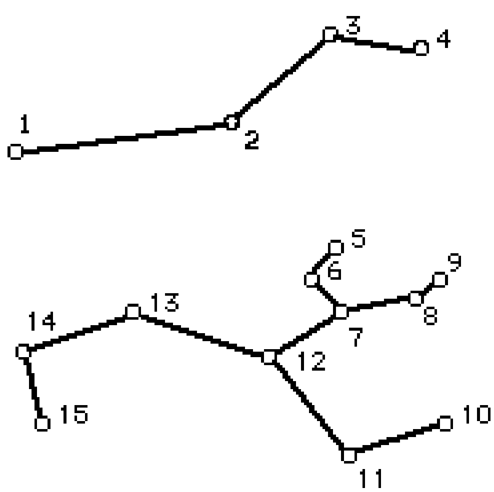
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Adding edge (10,12) would form a cycle, as would edge (2,4):



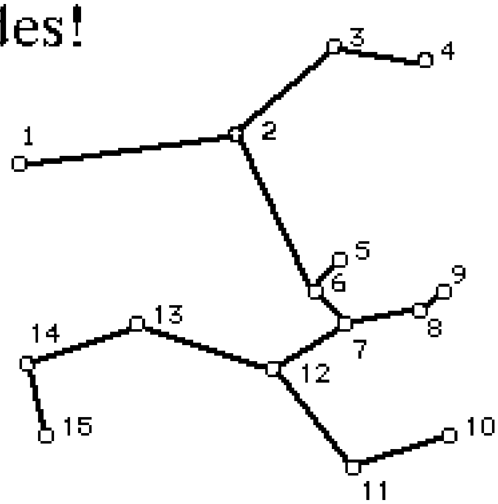
i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

The next edge to be added is (1,2), which leaves us with only two trees!



i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57

Finally, adding edge (2,6) leaves us with a single tree, spanning all of the nodes!



i	j	d_{ij}
9	8	6
5	6	8
7	6	9
8	7	12
5	7	14
3	4	15
7	12	16
15	14	16
10	11	17
5	9	18
12	6	18
14	13	20
12	13	24
3	2	25
12	11	26
13	6	30
12	10	33
4	2	35
1	2	36
6	2	37
12	15	40
4	5	46
11	15	50
1	3	57