Mathematical Programming Models of TSP

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author

Minimize $\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$

subject to

$$\sum_{j=1}^{n} X_{ij} = 1 \text{ for } i=1, \ldots n$$

one city must follow city i

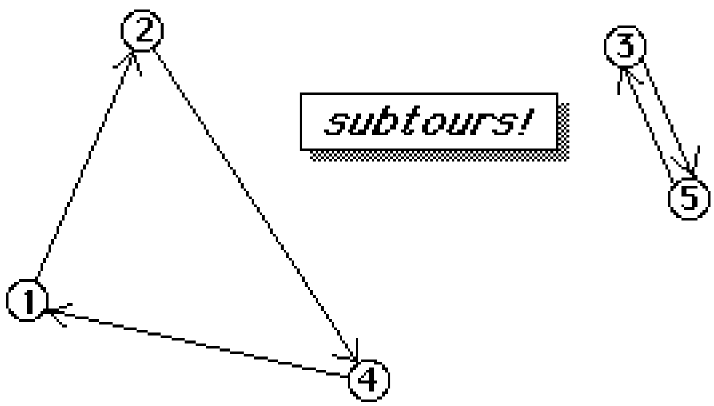
$$\sum_{i=1}^{n} X_{ij} = 1 \text{ for } j=1, \ldots n$$

one city must precede city j

$$X_{ij} \; \varepsilon \; \{0,1\} \text{ for all } i,j$$

These are the constraints of the assignment problem!

Not all feasible solutions of the assignment problem (AP) are TSP tours!

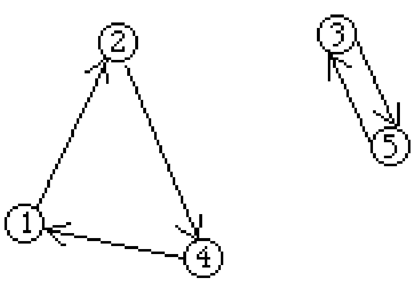


What constraints can be added to AP in order to eliminate the subtours?

- Introduce new variables u_i, i=1, 2, ...n
- For each pair of cities (i,j), $i \neq j$, add the constraint

$$u_i - u_j + nX_{ij} \leq n - 1$$

- These constraints will eliminate subtours.
- For example: $X_{12} = X_{24} = X_{41} = 1 \Rightarrow$



$$u_1 - u_2 + 5 \leq 4$$
$$u_2 - u_4 + 5 \leq 4$$
$$u_4 - u_1 + 5 \leq 4$$

sum: $15 \leq 12$ **infeasible!**

Reference: Miller, Tucker, & Zemlin, J.ACM, Vol. 7 (1960), pp.326ff.

These new constraints eliminate subtours of fewer than n cities, but NOT tours of n cities:

Let u_i = sequence # in which city i is visited.

So, if $X_{ij} = 1$, we have $u_i - u_j = -1$

and so $u_i - u_j + nX_{ij} = -1 + n$

i.e., $u_i - u_j + nX_{ij} \leq n - 1$ is satisfied!

Dimensions of model:

n(n-1)	integer variables X_{ij}
n	continuous variables u_i
2n	assignment constraints
n(n-1)	subtour elimination constraints

n (#cities)	5	10	50	100
variables				
0-1 integer	20	90	2450	9900
continuous	5	10	50	100
constraints	30	110	2550	10100

Another set of subtour elimination constraints:

- Let S be a nontrivial subset of the cities.

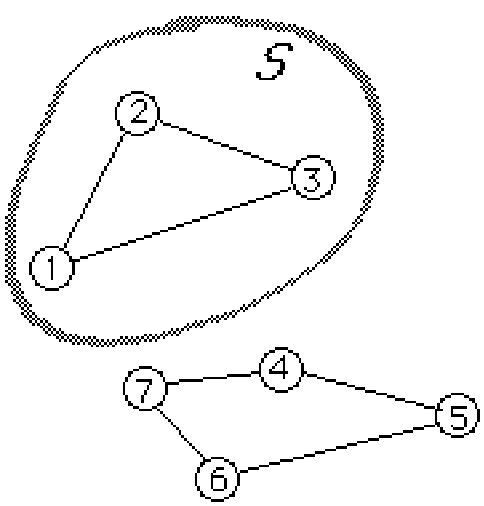
$$\sum_{i \notin S} \sum_{j \in S} X_{ij} \geq 1$$

insures that there is an edge in the tour which links set S to the set of cities NOT in S.

- If we include such a constraint for every nontrivial subset, we eliminate all subtours.

- For example, if S is the set of cities on a subtour, the constraint is violated!

Reference: Dantzig, Fulkerson, & Johnson, O.R. Vol. 7 (1959),pp.58ff.



$$\sum_{i \notin S} \sum_{j \in S} X_{ij} \geq 1$$

The subtour elimination constraint is violated, since

$$X_{41}=X_{51}=X_{61}=X_{71}=X_{42}=X_{52}=X_{62}=X_{72}=X_{43}=X_{53}=X_{63}=X_{73}=0$$

Unfortunately, the number of such constraints is exponential in the number of cities ($2^n - 1$):

n (#cities)	5	10	50	100
subtour elimination constraints	31	1024	1.126×10^{15}	1.268×10^{30}

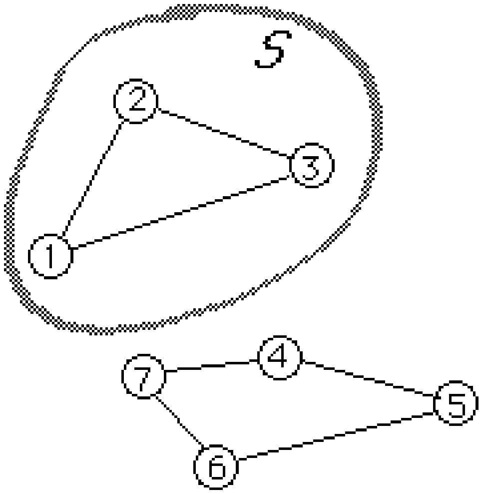
Another set of subtour elimination constraints

For every nontrivial subset S,

$$\sum_{i\in S}\sum_{j\in S} X_{ij} \le |S| - 1$$

where $|S|$ is the cardinality of the set S

The number of such constraints is again $2^n - 1$



$$\sum_{i \in S} \sum_{j \in S} X_{ij} \leq |S| - 1 = 2$$

The subtour elimination constraint is violated, since there are three edges in the subtour

Summary: Subtour Elimination Constraints

(to be appended to the assignment problem)

$$\sum_{i \notin S} \sum_{j \in S} X_{ij} \geq 1$$

for all subsets S of the nodes

$$\sum_{i \in S} \sum_{j \in S} X_{ij} \leq |S| - 1$$

for all subsets S of the nodes

$$u_i - u_j + nX_{ij} \leq n - 1$$

for all edges (i,j)

Warning!

While subtours are eliminated by any of the three sets of constraints,

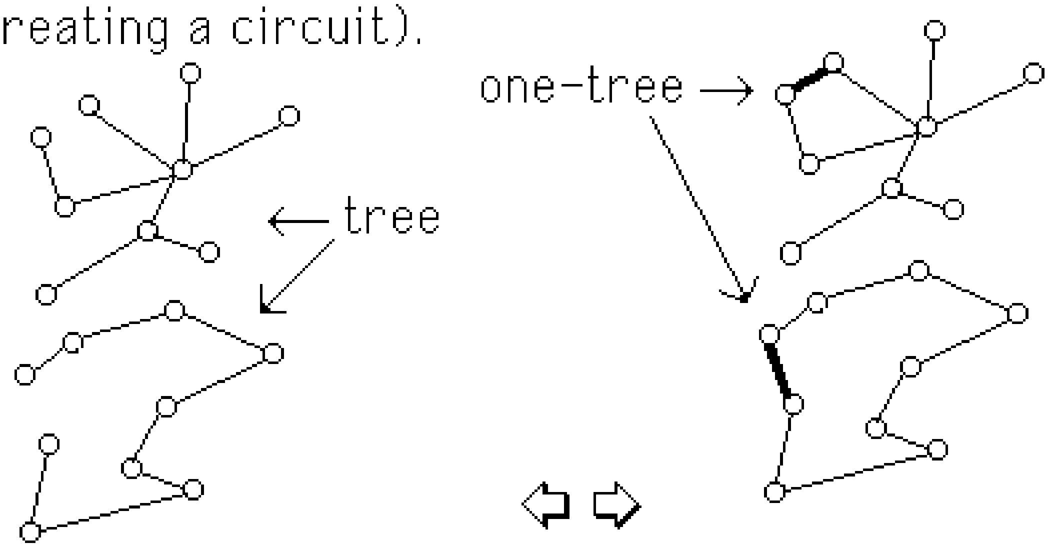
the smaller set of constraints, i.e.,

$$u_i - u_j + nX_{ij} \le n - 1$$ for all edges (i,j)

yields a much ***weaker*** lower bound in an LP relaxation!

One-Tree Constraint

A TSP tour is a special case of a "one-tree", which is a spanning tree with one additional edge included (creating a circuit).

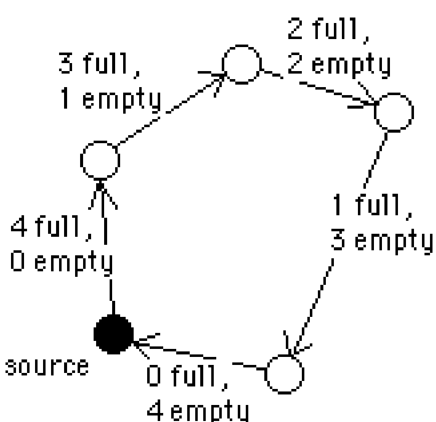


Let $\mathcal{T}_1$ denote the set of all one-trees of the network, so that $X \in \mathcal{T}_1$ if X is a one-tree.
Then the constraint

$$\boxed{X \in \mathcal{T}_1}$$

will eliminate subtours, since spanning one-trees are connected!

2-commodity flow model



Suppose that the salesman delivers a full container of some commodity (for example, bottled gas), and picks up an empty container, for each customer.

At all times, he will have a total of (n-1) containers (full plus empty) in his vehicle.

Define **two** sets of **continuous** variables (one for flow of full containers, and one for flow of empty containers), plus one set of **binary** variables.

Y^P_{ij} = flow of full containers in edge (i,j)

Y^Q_{ij} = flow of empty containers in edge (i,j)

$$X_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ is on the route} \\ 0 & \text{otherwise} \end{cases}$$

Constraints

(These guarantee a path from source to each node, & a return path, eliminating subtours.)

$$\sum_{j=1}^{n} Y^P_{ij} - \sum_{k=1}^{n} Y^P_{ki} = \begin{cases} n-1 & \text{for } i = \text{source} \\ -1 & \text{elsewhere} \end{cases}$$

$Y^P_{ij} \geq 0$ ***conservation of flow for full containers***

$$\sum_{j=1}^{n} Y^Q_{ij} - \sum_{k=1}^{n} Y^Q_{ki} = \begin{cases} -(n-1) & \text{for } i = \text{source} \\ +1 & \text{elsewhere} \end{cases}$$

$Y^Q_{ij} \geq 0$ ***conservation of flow for empty containers***

Since the total flow in each edge of the tour is $n-1$, we also have:

$$Y^P_{ij} + Y^Q_{ij} = (n-1)\,X_{ij} \quad \text{for each edge } (i,j)$$

That is, if $X_{ij}=0$, no flow is permitted in edge (i,j)

while if $X_{ij}=1$, the total flow is $n-1$

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Objective

$$\text{Minimize} \quad \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} + \beta/_{n-1} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \left(Y^P_{ij} + Y^Q_{ij} \right)$$

where $\alpha + \beta = 1$, $\alpha \geq 0$ & $\beta \geq 0$

For example ($\alpha=1, \beta=0$): $\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$

or ($\alpha=0, \beta=1$): $\text{Minimize} \frac{1}{(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \left(Y^P_{ij} + Y^Q_{ij} \right)$

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This model easily incorporates

Precedence constraints

Suppose city **h** must precede city **k** on the tour. Then the number of "full containers" entering city **h** must exceed the number of "full containers" entering city **k**:

$$\sum_{i=1}^{n} Y_{ih}^{P} \geq 1 + \sum_{i=1}^{n} Y_{ik}^{P}$$

(n-1) commodity flow model

For each city **k** (other than the source **s**), we define a commodity:

Y_{ij}^{k} = flow in arc **(i,j)** of commodity destined for **k**

Constraints

Conservation of flow for each commodity k:

$$\sum_{j=1}^{n} Y_{sj}^{k} = 1 \qquad \text{at source } s$$

$$\sum_{i=1}^{n} Y_{ih}^{k} = \sum_{j=1}^{n} Y_{hj}^{k} \qquad \text{at } h \neq s,\ h \neq k$$

$$\sum_{i=1}^{n} Y_{ik}^{k} = 1 \ \ \& \ \sum_{j=1}^{n} Y_{kj}^{k} = 0 \qquad \text{at destination } k$$

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Capacity constraints

$$0 \le Y_{ij}^{k} \le X_{ij} \qquad \text{for each arc } (i,j)$$

Assignment constraints

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad \text{for all } i$$

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad \text{for all } j$$

$$X_{ij} \in \{0,1\} \qquad \text{for all } i\&j$$

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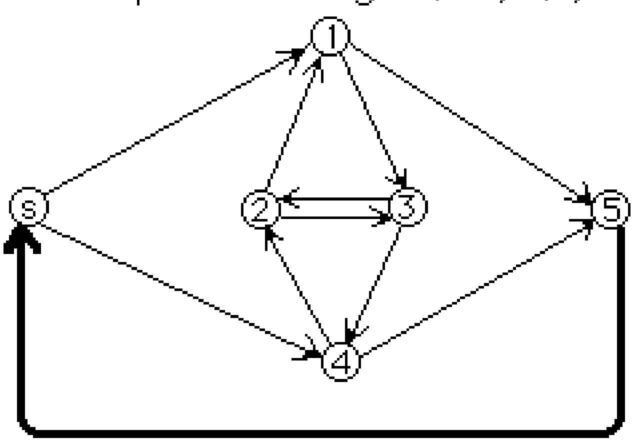
The feasible region of the LP relaxation of the (n-1)-commodity flow model is the same as that of the model with exponentially many subtour elimination constraints!

However, basic feasible solutions of this LP may be fractional, as in the following example.

Suppose that $X_{ij} = \frac{1}{2}$ for each edge below, except for edge (5,s), where $X_{5,s} = 1$.



Then X is a basic feasible solution to the LP relaxation of the assignment & subtour elimination constraints.

Can constraints be added to eliminate this solution?

"Mutual Flow Constraints"

The following n(n-1)/2 constraints will eliminate the previous solution:

$$\sum_{i=1}^{n} Y_{ik}^{j} + \sum_{i=1}^{n} Y_{ij}^{k} = 1$$ for each pair j,k ($j \neq k$)

(either commodity j flows through node k, or commodity k flows through node j, but not both!)

Author Dantzig, G.B., Fulkerson, D.R., and Johnson, S.M.

Title Solutions of a large scale traveling salesman problem

Pub. Operations Research, Volume 2 (1954), pp. 393-410

Notes

Key subtour elimination constraints

Author Miller, C.E., Tucker, A.W., and Zemlin, R.A.

Title Integer programming formulation of traveling salesman problems

Pub. J. ACM, Volume 7 (1960), pp. 326-329

Notes n(n-1)/2 subtour elimination constraints

Key subtour elimination constraints

Author Kusiak, Andrew and Finke, Gerd

Title Modeling and solving the flexible forging module scheduling problem

Pub. Engineering Optimization, Volume 12 (1987), pp. 1-12

Notes

Key 2-commodity flow model of TSP

Author Wong, Richard T.

Title Integer programming formulations of the traveling salesman problem

Pub. Proceedings of IEEE International Conference on Circuits & Computers, 1980

Notes

Key multi-commodity flow model

TSP.models

Author Claus, A.

Title A new formulation for the travelling salesman problem

Pub. SIAM J. Alg. Disc. Meth., Volume 5, Number 1 (March 1984), pp. 21-25

Notes

Key multi-commodity flow model