Benders' Decomposition Applied to Stochastic Linear Programming with Recourse

Stochastic LP

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Deterministic Equivalent LP for the 2-stage Stochastic LP with Recourse

$$
\begin{array}{llllll}
\text{Minimize} & cx + p_1 dy^1 + p_2 dy^2 + \dots + p_k dy^k & & & & \\
\text{subject to} & Ax + By^1 & & & & = b^1 \\
& Ax & + By^2 & & & = b^2 \\
& Ax & & + By^3 & & = b^3 \\
& \vdots & & & \ddots & \vdots \\
& Ax & & & + By^k & = b^k \\
& & & & & \\
& x \geq 0, y^1 \geq 0, \dots, y^k \geq 0 & & & &
\end{array}
$$

First-stage cost
plus
expected
2^{nd}-stage cost

This can be a truly huge LP to solve!
For example, suppose that b consists of 10 RHSs, with each RHS having 3 possible values. Then k, the number of scenarios, is $3^{10} = 59{,}049$ while the number of rows in the LP is 10 times that, nearly 600,000!

$$
\begin{array}{llllll}
\text{Minimize} & cx + p_1 dy^1 + p_2 dy^2 + \cdots + p_k dy^k & & & & \\
\text{subject to} & Ax + By^1 & & & & = b^1 \\
 & Ax & + By^2 & & & = b^2 \\
 & Ax & & + By^3 & & = b^3 \\
 & \vdots & & & \ddots & \vdots \\
 & Ax & & & + By^k & = b^k \\
 & x \ge 0,\ y^1 \ge 0, \ldots, y^k \ge 0 & & & &
\end{array}
$$

This is beyond the limitations of most LP solvers.

Benders' algorithm partitions the variables into two sets...

in this case,

- the 1st-stage variables x (which will be selected by the "master problem")

and

- the 2nd-stage variables $y^1, y^2, \ldots y^k$ (which will be assigned optimal values by the subproblem, given a trial 1st stage solution by the master problem.

The Master Problem is to minimize $v(x)$, subject to $x \geq 0$, where

$$v(x) = c\,x + \sum_{i=1}^{k} \left\{ \begin{array}{l} \text{minimum } p_i\, d\, y^i \\ \text{subject to} \\ \qquad B y^i \leq b^i - A x \\ \qquad y^i \geq 0 \end{array} \right\}$$

That is, for any x, we evaluate $v(x)$ by adding the 1st-stage cost cx to the sum of the 2nd-stage LP solutions!

By LP duality theory,

$$v(x) = c\,x + \sum_{i=1}^{k} p_i \left\{ \begin{array}{ll} \text{minimum} & d\,y^i \\ \text{subject to} & B y^i \leq b^i - Ax \\ & y^i \geq 0 \end{array} \right\}$$

is equivalent to

$$v(x) = c\,x + \sum_{i=1}^{k} p_i \left\{ \begin{array}{ll} \text{maximum} & (b^i - Ax)U^i \\ \text{subject to} & B^t U^i \leq d\,y^i \\ & U^i \geq 0 \end{array} \right\}$$

That is,

$$v(x) = \sum_{i=1}^{k} p_i \left\{ \begin{array}{ll} \text{maximum} & c\,x + (b^i - Ax)U^i \\ \text{subject to} & B^t U^i \leq d\,y^i \\ & U^i \geq 0 \end{array} \right\}$$

Objective

Consider again the stochastic problem of the golf-bag manufacturer (Par, Inc.)

$$\text{Max } 10X_1 + 9X_2 + 0.3\left(8Y_1^0 - 5T_{CD}^0 - 6T_S^0 - 8T_F^0 - 4T_{IP}^0\right)$$
$$+\ 0.3\left(8Y_1^1 - 5T_{CD}^1 - 6T_S^1 - 8T_F^1 - 4T_{IP}^1\right)$$
$$+\ 0.3\left(8Y_1^2 - 5T_{CD}^2 - 6T_S^2 - 8T_F^2 - 4T_{IP}^2\right)$$
$$+ 0.1\left(8Y_1^3 - 5T_{CD}^3 - 6T_S^3 - 8T_F^3 - 4T_{IP}^3\right)$$

Equivalent Deterministic Linear Programming Model

The objective function of the deterministic equivalent LP, using the dual subproblems, is:

$$
\begin{aligned}
&10X_1 + 9X_2 \\
&+ 0.3\left[(630\text{-}0.7X_1\text{-}X_2)U_1^0 + \cdots + (135\text{-}0.1X_1\text{-}0.25X_2)U_4^0 + 100U_5^0)\right] \\
&+ 0.3\left[(580\text{-}0.7X_1\text{-}X_2)U_1^1 + \cdots + (125\text{-}0.1X_1\text{-}0.25X_2)U_4^1 + 100U_5^1)\right] \\
&+ 0.3\left[(600\text{-}0.7X_1\text{-}X_2)U_1^2 + \cdots + (120\text{-}0.1X_1\text{-}0.25X_2)U_4^2 + 100U_5^2)\right] \\
&+ 0.1\left[(550\text{-}0.7X_1\text{-}X_2)U_1^3 + \cdots + (110\text{-}0.1X_1\text{-}0.25X_2)U_4^3 + 100U_5^3)\right]
\end{aligned}
$$

$$
\text{.... or, Max}\;
\begin{bmatrix}
10 - 0.3\left(0.7U_1^0 + 0.5U_2^0 + U_3^0 + 0.1U_4^0\right) \\
- 0.3\left(0.7U_1^1 + 0.5U_2^1 + U_3^1 + 0.1U_4^1\right) \\
- 0.3\left(0.7U_1^2 + 0.5U_2^2 + U_3^2 + 0.1U_4^2\right) \\
- 0.1\left(0.7U_1^3 + 0.5U_2^3 + U_3^3 + 0.1U_4^3\right)
\end{bmatrix} X_1
$$

$$
+ \begin{bmatrix}
9 - 0.3\left(U_1^0 + 0.8333U_2^0 + 0.6667U_3^0 + 0.25U_4^0\right) \\
- 0.3\left(U_1^1 + 0.8333U_2^1 + 0.6667U_3^1 + 0.25U_4^1\right) \\
- 0.3\left(U_1^2 + 0.8333U_2^2 + 0.6667U_3^2 + 0.25U_4^2\right) \\
- 0.1\left(U_1^3 + 0.8333U_2^3 + 0.6667U_3^3 + 0.25U_4^3\right)
\end{bmatrix} X_2
$$

$$
+ \begin{bmatrix}
0.3\left(630U_1^0 + 600U_2^0 + 708U_3^0 + 135U_4^0 + 100U_5^0\right) \\
+ 0.3\left(580U_1^1 + 560U_2^1 + 628U_3^1 + 125U_4^1 + 100U_5^1\right) \\
+ 0.3\left(600U_1^2 + 550U_2^2 + 638U_3^2 + 120U_4^2 + 100U_5^2\right) \\
+ 0.1\left(550U_1^3 + 510U_2^3 + 558U_3^3 + 110U_4^3 + 100U_5^3\right)
\end{bmatrix}
$$

Linear in X, ⇦ ⇨ for fixed values of U!

Benders' algorithm alternates between

"master" problem

and

"subproblem",

and can initially be started with either.

In this case, we will begin with the subproblem, which requires a "guess" of the optimal first-stage variables (X).

Each time we solve the subproblem (which separates into an LP for each scenario), we will obtain dual variables which allow us to compute a linear approximation to the master problem objective, v(X).

To get an initial trial solution for our stage-1 variables, we arbitrarily use the optimal scheule for scenario 3, i.e., assuming both bids are successful, and the capacities in the various departments are reduced accordingly.

```
MAX  10X1 + 9 X2
st
0.7X1 + X2              <550
0.5X1 + 0.83333X2       <510
   X1 + 0.66667X2       <558
0.1X1 + 0.25X2          <110
END
```

OBJECTIVE FUNCTION VALUE

6269.810

VARIABLE	VALUE	REDUCED COST
X1	360.9077	.0000
X2	295.6369	.0000

That is, our initial trial solution (stage 1) is

X1	360.907740
X2	295.636900

or

schedule production of

360.1 standard golf bags

and 295.6 deluxe golf bags

Now, using the trial values of X, namely

X1	360.907740
X2	295.636900

we solve the second-stage problem for each of the four scenarios:

Scenario #0	Company fails to obtain both contracts
Scenario #1	Company wins contract #1, loses #2
Scenario #2	Company wins contract #2, loses #1
Scenario #3	Company wins both contracts #1 & #2

Click to obtain solution for each scenario!

SCENARIO 0 company loses both contracts

In this case, the right-hand-sides (the capacities in the various departments) are the original departmental capacities, minus the amounts of the capacities used by the trial stage-1 production schedule of 360.9 standard and 295.6 deluxe golf bags.

SCENARIO 0 company loses both contracts

```
MAX    8 Y10 - 5 TCD0 - 6 TS0
              - 8 TF0 - 4 TIP0
 SUBJECT TO
      2)  0.7 Y10 - TCD0 <=  81.723
      3)  0.5 Y10 - TS0  <= 173.178
      4)  Y10 - TF0      <= 149.997
      5)  0.1 Y10 - TIP0 <=  25
      6)               TF0 <= 100
 END
```

OBJECTIVE FUNCTION VALUE

1083.601

VARIABLE	VALUE	REDUCED COST
Y0	149.9970	.0000
TF0	.0000	6.0000
TS0	.0000	3.5000
TIP0	.0000	4.0000
TCD0	23.2748	.0000

SCENARIO 0

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.0000	5.0000
3)	98.1795	.0000
4)	.0000	4.5000
5)	10.0003	.0000
6)	100.0000	.0000

SCENARIO 0

RECOURSE

That is, if the initial trial solution

$$X = (360.9, 295.6)$$

were chosen, and scenario 0 occurs, i.e., neither bid was successful, then some capacity remains idle in each of the departments... the optimal recourse would be to schedule production of an additional 98.2 standard golf bags and an additional 23.275 hours in the cutting & dyeing department.

SCENARIO 0

SCENARIO 1 company wins 1st contract, loses 2nd contract

```
MAX     8 Y11 - 5 TCD1 - 6 TS1
              - 8 TF1 - 4 TIP1
SUBJECT TO
     2)  0.7 Y11 - TCD1 <=  31.723
     3)  0.5 Y11 - TS1  <= 133.178
     4)  Y11 - TF1      <=  69.997
     5)  0.1 Y11 - TIP1 <=  15
     6)            TF1  <= 100
END
```

OBJECTIVE FUNCTION VALUE

473.6015

VARIABLE	VALUE	REDUCED COST
Y1	69.9970	.0000
TCD1	17.2749	.0000
TS1	.0000	6.0000
TF1	.0000	3.5000
TIP1	.0000	4.0000

SCENARIO 1

RECOURSE

That is, if the trial stage-1 solution

$$X = (360.9, 295.6)$$

were used, and the first bid (but not the second) were successful, each of the four departments will have excess capacity...
the optimal recourse is to schedule production of an additional 69.997 standard golf bags, and use of 17.275 hours of overtime in the cutting & dyeing department.

SCENARIO 1

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.0000	5.0000
3)	98.1794	.0000
4)	.0000	4.5000
5)	8.0002	.0000
6)	100.0000	.0000

⏎

SCENARIO 1

SCENARIO 2 company loses 1st contract, wins 2nd contract

```
MAX    8 Y12 - 5 TCD2 - 6 TS2
             - 8 TF2  - 4 TIP2
SUBJECT TO
      2)  0.7 Y12 - TCD2 <=  51.723
      3)  0.5 Y12 - TS2 <=  123.178
      4)  Y12 - TF2 <=  79.997
      5)  0.1 Y12 - TIP2 <=  10
      6)  TF2 <=  100
END
```

↵

OBJECTIVE FUNCTION VALUE

618.6015

VARIABLE	VALUE	REDUCED COST
Y2	79.9970	.0000
TCD2	4.2749	.0000
TS2	.0000	6.0000
TF2	.0000	3.5000
TIP2	.0000	4.0000

SCENARIO 2

SCENARIO 2 company loses 1st contract, wins 2nd contract

That is, the company should

- schedule production of additional 80 standard golf bags
- schedule 4.275 hours of overtime in the Cutting & Dyeing department

RECOURSE

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.0000	5.0000
3)	83.1795	.0000
4)	.0000	4.5000
5)	2.0003	.0000
6)	100.0000	.0000

↵

SCENARIO 2

SCENARIO 3 company wins both contracts!

```
MAX    8 Y13 - 5 TCD3 - 6 TS3
             - 8 TF3   - 4 TIP3
SUBJECT TO
     2)  0.7 Y3  - TCD3 <=   1.723
     3)  0.5 Y3  - TS3  <=  83.178
     4)  Y3  - TF3      <=   0
     5)  0.1 Y3  - TIP3 <=   0
     6)            TF3  <= 100
END
```

⏎

OBJECTIVE FUNCTION VALUE

.0000000

VARIABLE	VALUE	REDUCED COST
Y13	.0000	.0000
TCD3	.0000	5.0000
TS3	.0000	6.0000
TF3	.0000	.0000
TIP3	.0000	4.0000

SCENARIO 3

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	1.7230	.0000
3)	83.1780	.0000
4)	.0000	8.0000
5)	.0000	.0000
6)	100.0000	.0000

SCENARIO 3

SCENARIO 3 company wins both contracts!

If the trial stage-1 solution X=(360.9, 295.6) were used, and the third scenario occurs, i.e., both bids are successful...
then the optimal recourse is neither to add production of the standard bags nor to schedule overtime hours in any department.

RECOURSE ⮐

The expected profit *if* X=(360.9, 295.6) were selected as the values of the stage-1 variables, is

profit	
6269.81	1st stage
0.3x1083.6	scenario 0
0.3x 473.6	scenario 1
0.3x 618.6	scenario 2
0.1x 0	scenario 3
6922.6	TOTAL

DUAL PRICES

The optimal dual variables from the subproblem will be used to compute a linear approximation to v(X)

	scenario			
	0	1	2	3
U_1	5.0000	5.0000	5.0000	.0000
U_2	.0000	.0000	.0000	.0000
U_3	4.5000	4.5000	4.5000	8.0000
U_4	.0000	.0000	.0000	.0000
U_5^0	.0000	.0000	.0000	.0000

$$
\begin{aligned}
\text{Max} &\left[\begin{array}{l}
10 - 0.3\left(0.7U_1^0 + 0.5U_2^0 + U_3^0 + 0.1U_4^0\right) \\
\quad - 0.3\left(0.7U_1^1 + 0.5U_2^1 + U_3^1 + 0.1U_4^1\right) \\
\quad - 0.3\left(0.7U_1^2 + 0.5U_2^2 + U_3^2 + 0.1U_4^2\right) \\
\quad - 0.1\left(0.7U_1^3 + 0.5U_2^3 + U_3^3 + 0.1U_4^3\right)
\end{array}\right] X_1 \\
+ &\left[\begin{array}{l}
9 - 0.3\left(U_1^0 + 0.8333U_2^0 + 0.6667U_3^0 + 0.25U_4^0\right) \\
\quad - 0.3\left(U_1^1 + 0.8333U_2^1 + 0.6667U_3^1 + 0.25U_4^1\right) \\
\quad - 0.3\left(U_1^2 + 0.8333U_2^2 + 0.6667U_3^2 + 0.25U_4^2\right) \\
\quad - 0.1\left(U_1^3 + 0.8333U_2^3 + 0.6667U_3^3 + 0.25U_4^3\right)
\end{array}\right] X_2 \\
+ &\left[\begin{array}{l}
0.3\left(630U_1^0 + 600U_2^0 + 708U_3^0 + 135U_4^0 + 100U_5^0\right) \\
+ 0.3\left(580U_1^1 + 560U_2^1 + 628U_3^1 + 125U_4^1 + 100U_5^1\right) \\
+ 0.3\left(600U_1^2 + 550U_2^2 + 638U_3^2 + 120U_4^2 + 100U_5^2\right) \\
+ 0.1\left(550U_1^3 + 510U_2^3 + 558U_3^3 + 110U_4^3 + 100U_5^3\right)
\end{array}\right]
\end{aligned}
$$

We now compute this objective for fixed values of U!

Substituting the dual variables which optimize the subproblems, we obtain a linear function in the first-stage variables, X_1 and X_2 :

$$2\,X_1 + 1.2667\,X_2 + 5826.3$$

☞ *Note: Evaluating the linear function at X^0 = (360.9, 295.6), the fixed values of the first-stage problem, gives 6922.6, which agrees with the total expected profit from stages 1 & 2 found earlier! That is, this "approximation" is exact at X^0.*

We now have our first linear approximation of the optimal value function v(X):

$$2\,X_1 + 1.2667\,X_2 + 5826.3$$

☞ *Note: this is an "upper approximation" of v(X)!*

The "master problem" is now solved, to maximize this approximation of v(X), subject to any restrictions we might want to place on X.

Master Problem

$$\text{Max} \quad 2X_1 + 1.2667X_2 + 5826.3$$

$$\begin{aligned}
\text{subject to} \quad \tfrac{7}{10}X_1 + X_2 &\le 630 \\
\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 &\le 600 \\
X_1 + \tfrac{2}{3}X_2 &\le 708 \\
\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 &\le 135 \\
X_1 \ge 0,\ X_2 &\ge 0
\end{aligned}$$

Without any constraints, the optimal master problem solution will be unbounded above, i.e., both X_1 and X_2 → infinity

We should therefore restrict X in some way... here, we require that X will not use more than the maximum possible # of hours in the 4 depts.

OBJECTIVE FUNCTION VALUE

1416.000

VARIABLE	VALUE	REDUCED COST
X1	708.00	.0000
X2	.00	.0666

Right-Hand-Sides of 2nd Stage Problems

row	scenario 0	scenario 1	scenario 2	scenario 3
1	134.4	84.4	104.4	54.4
2	246	206	196	156
3	0	-80	-70	-150
4	64.2	54.2	49.2	39.2
5	100	100	100	100

Now, using the trial values of X, namely

X1	708.00
X2	.00

we solve the second-stage problem for each of the four scenarios:

Scenario #0	Company fails to obtain both contracts
Scenario #1	Company wins contract #1, loses #2
Scenario #2	Company wins contract #2, loses #1
Scenario #3	Company wins both contracts #1 & #2

Click to obtain solution for each scenario!

SCENARIO 0 company loses both contracts

```
MAX    8 Y0  - 5 TCD0 - 6 TS0
             - 8 TF0 - 4 TIP0 - 100 ART
SUBJECT TO
     2)  0.7 Y0  - TCD0 - ART <=  134.4
     3)  0.5 Y0  - TS0  - ART <=    0
     4)  Y0  - TF0      - ART <=   64.2
     5)  0.1 Y0  - TIP0 - ART <=   45
     6)  TF0                  <=  100
END
```

↵

OBJECTIVE FUNCTION VALUE

1) 321.0000

VARIABLE	VALUE	REDUCED COST
Y0	64.200000	.000000
TCD0	.000000	5.000000
TS0	32.100000	.000000
TF0	.000000	3.000000
TIP0	.000000	4.000000
ART	.000000	89.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	89.460000	.000000
3)	.000000	6.000000
4)	.000000	5.000000
5)	38.580001	.000000
6)	100.000000	.000000

SCENARIO 1

company wins 1st contract,
loses 2nd contract

```
MAX     8 Y1   - 5 TCD1 - 6 TS1
               - 8 TF1 - 4 TIP1 - 100 ART
SUBJECT TO
     2)  0.7 Y1   - TCD1 - ART <=    84.4
     3)  0.5 Y1   - TS1  - ART <=   206
     4)  Y1   - TF1      - ART <=  - 80
     5)  0.1 Y1   - TIP1 - ART <=    54.2
     6)  TF1                   <=   100
END
```

⏎

OBJECTIVE FUNCTION VALUE

1) -640.0000

VARIABLE	VALUE	REDUCED COST
Y1	.000000	.000000
TCD1	.000000	5.000000
TS1	.000000	6.000000
TF1	80.000000	.000000
TIP1	.000000	4.000000
ART	.000000	92.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	84.400001	.000000
3)	206.000000	.000000
4)	.000000	8.000000
5)	54.200000	.000000
6)	20.000000	.000000

SCENARIO 2

company loses 1st contract,
wins 2nd contract

```
MAX     8 Y2   - 5 TCD2 - 6 TS2
               - 8 TF2 - 4 TIP2 - 100 ART
SUBJECT TO
     2)  0.7 Y2   - TCD2 - ART <=   104.4
     3)  0.5 Y2   - TS2  - ART <=   196
     4)  Y2  - TF2       - ART <=   - 70
     5)  0.1 Y2   - TIP2 - ART <=    49.2
     6)  TF2                   <=   100
END
```

↵

OBJECTIVE FUNCTION VALUE

1) -560.0000

VARIABLE	VALUE	REDUCED COST
Y2	.000000	.000000
TCD2	.000000	5.000000
TS2	.000000	6.000000
TF2	70.000000	.000000
TIP2	.000000	4.000000
ART	.000000	92.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	104.400000	.000000
3)	196.000000	.000000
4)	.000000	8.000000
5)	49.200000	.000000
6)	30.000000	.000000

SCENARIO 3 company wins both contracts!

```
MAX     8 Y3   - 5 TCD3 - 6 TS3
               - 8 TF3 - 4 TIP3 - 100 ART
SUBJECT TO
      2)  0.7 Y3   - TCD3 - ART <=    54.4
      3)  0.5 Y3   - TS3  - ART <=   156
      4)  Y3   - TF3      - ART <= - 150
      5)  0.1 Y3   - TIP3 - ART <=    39.2
      6)  TF3                   <=   100
END
```

OBJECTIVE FUNCTION VALUE

1) -5800.000

VARIABLE	VALUE	REDUCED COST
Y3	.000000	92.000000
TCD3	.000000	5.000000
TS3	.000000	6.000000
TF3	100.000000	.000000
TIP3	.000000	4.000000
ART	50.000000	.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	104.400000	.000000
3)	206.000000	.000000
4)	.000000	100.000000
5)	89.200000	.000000
6)	.000000	92.000000

The expected profit *if* X=(708, 0) were selected as the values of the stage-1 variables, is

profit	
7080	1st stage
0.3x 321.00	scenario 0
0.3x ⁻640.00	scenario 1
0.3x ⁻560.00	scenario 2
0.1x⁻5800.00	scenario 3
6236.3	TOTAL

DUAL PRICES

The optimal dual variables from the subproblem will be used to compute a linear approximation to v(X)

	scenario			
	0	1	2	3
U_1	.000	.000	.000	.000
U_2	6.000	.000	.000	.000
U_3	5.000	8.000	8.000	100.000
U_4	.000	.000	.000	.000
U_5^0	.000	.000	.000	92.000

$$
\text{Max}\begin{bmatrix} 10 - 0.3(0.7U_1^0 + 0.5U_2^0 + U_3^0 + 0.1U_4^0) \\ - 0.3(0.7U_1^1 + 0.5U_2^1 + U_3^1 + 0.1U_4^1) \\ - 0.3(0.7U_1^2 + 0.5U_2^2 + U_3^2 + 0.1U_4^2) \\ - 0.1(0.7U_1^3 + 0.5U_2^3 + U_3^3 + 0.1U_4^3) \end{bmatrix} X_1
+ \begin{bmatrix} 9 - 0.3(U_1^0 + 0.8333U_2^0 + 0.6667U_3^0 + 0.25U_4^0) \\ - 0.3(U_1^1 + 0.8333U_2^1 + 0.6667U_3^1 + 0.25U_4^1) \\ - 0.3(U_1^2 + 0.8333U_2^2 + 0.6667U_3^2 + 0.25U_4^2) \\ - 0.1(U_1^3 + 0.8333U_2^3 + 0.6667U_3^3 + 0.25U_4^3) \end{bmatrix} X_2
$$

$$
+ \begin{bmatrix} 0.3\,(630U_1^0 + 600U_2^0 + 708U_3^0 + 135U_4^0 + 100U_5^0) \\ + 0.3\,(580U_1^1 + 560U_2^1 + 628U_3^1 + 125U_4^1 + 100U_5^1) \\ + 0.3\,(600U_1^2 + 550U_2^2 + 638U_3^2 + 120U_4^2 + 100U_5^2) \\ + 0.1\,(550U_1^3 + 510U_2^3 + 558U_3^3 + 110U_4^3 + 100U_5^3) \end{bmatrix}
$$

We now compute this objective for fixed values of U!

Substituting the dual variables which optimize the subproblems, we obtain a linear function in the first-stage variables, X_1 and X_2 :

$$-7.2\,X_1 - 3.3667\,X_2 + 11680$$

☞ *Note: Evaluating the linear function at $X^1 = (708, 0)$, the fixed values of the first-stage problem, gives 6582.4 which agrees with the total expected profit from stages 1 & 2 found earlier! That is, this "approximation" is exact at X^1.*

no!

Master Problem

Max z
subject to

$$z \le 2X_1 + 1.2667X_2 + 5826.3$$
$$z \le -7.2X_1 - 3.3667X_2 + 11680$$
$$\tfrac{7}{10}X_1 + X_2 \le 630$$
$$\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 \le 600$$
$$X_1 + \tfrac{2}{3}X_2 \le 708$$
$$\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 \le 135$$
$$X_1 \ge 0,\ X_2 \ge 0$$

OBJECTIVE FUNCTION VALUE

1) 7172.817

VARIABLE	VALUE	REDUCED COST
X1	492.6717	.000000
X2	285.1298	.000000
Z	7172.8173	.000000

That is, our next trial solution is 492.67 standard and 285.13 deluxe bags. $7172.81 is an *upper* bound on the maximum profit!