SIGNOMIAL GEOMETRIC PROGRAMMING

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Signomial Function

differs from a POSYNOMIAL FUNCTION in that the coefficients need not be positive.

$$g_k(x) = \sum_{i\in[k]} \sigma_i c_i \prod_{j=1}^{m} x_j^{a_{ij}}$$

where c_i = absolute value of coefficient
σ_i = sign of coefficient (+1 or −1)

EXAMPLE

$$g_0(x) = 2x_1^2 x_2^{-1} - 5x_1x_2$$

For this signomial:

$$\sigma_1 = +1 \quad \sigma_2 = -1$$

$$c_1 = 2 \quad c_2 = 5$$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Primal Signomial GP Problem

$$\text{Minimize } \; g_0(x_1, x_2, \cdots x_m)$$

$$\text{subject to } \; g_k(x_1, x_2, \cdots x_m) \le \zeta_k, \; k = 1, 2, \cdots p$$

$$x_j > 0, \; j = 1, 2, \cdots m$$

where $$g_k(x) = \sum_{i \in [k]} \sigma_i c_i \prod_{j=1}^{m} x_j^{a_{ij}}$$

signs

$\sigma_i = \pm 1 \;\; \forall i$

$\zeta_k = \pm 1 \;\; \forall k$

EXAMPLE

$$\text{Minimize } \; -2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$$

$$\text{subject to } \; -x_4^{1/2} + x_2^{1/3}x_3 \le -1$$

$$x_1, x_2, x_3, x_4 > 0$$

$\sigma = [-1, +1, +1, -1, +1]$

$c = [2, \; 1, \; 5, \; 1, \; 1]$

$\zeta_0 = \; ?$ *(sign of objective at optimum)*

$\zeta_1 = -1$

$$A = \begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & -1 & -1 & 0 \\ 1/2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1 & 0 \end{bmatrix}$$

"pseudo" Dual Signomial GP Problem

Maximize $\zeta_0 \left[\prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\sigma_i \delta_i} \prod_{k=1}^{p} \lambda_k^{\zeta_k \lambda_k} \right]^{\zeta_0}$

subject to

$$\sum_{i \in [k]} \sigma_i \delta_i = \zeta_k \lambda_k, \; k=0, 1, \cdots p$$

$$\sum_{i=1}^{n} \sigma_i a_{ij} \delta_i = 0, \; j=1,2, \cdots m$$

$$\delta_i \geq 0, \; i=1, 2, \cdots n$$

$$\lambda_0 = 1$$

$$\lambda_k \geq 0, \; k=1,2, \cdots p$$

For every locally minimum $\hat{x}$ of the primal signomial GP problem, there exists a dual feasible solution $(\hat{\delta}, \hat{\lambda})$ and sign ζ_0 such that

$$g_0(\hat{x}) = v(\hat{\delta}, \hat{\lambda})$$

Furthermore, $\hat{x}$ and $(\hat{\delta}, \hat{\lambda})$ are related by

$$c_i \prod_j \hat{x}_j^{a_{ij}} = \zeta_0 \hat{\delta}_i g_0(\hat{x}) \quad \forall\, i \in [0]$$

$$c_i \prod_j \hat{x}_j^{a_{ij}} = \frac{\hat{\delta}_i}{\hat{\lambda}_k} \; \forall\, i \in [k], \; k \geq 1$$

A *Weak Duality* theorem would say that

$$g_0(x) \geq v(\delta, \lambda)$$

for any primal-feasible x
and dual-feasible δ, λ

This is *NOT* true of the pseudo-dual signomial GP problem!

EXAMPLE

Minimize $-2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$

subject to $-x_4^{1/2} + x_2^{1/3}x_3 \leq -1$

$x_1, x_2, x_3, x_4 > 0$

$$\textbf{Max} \quad \zeta_0\left[\left(\frac{2}{\delta_1}\right)^{-\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{5}{\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{-\delta_4}\left(\frac{1}{\delta_5}\right)^{\delta_5}\lambda_1^{-\lambda_1}\right]^{\zeta_0}$$

pseudo-dual objective

$$\delta_i \geq 0,\ i=1, \cdots 5;\ \lambda_1 \geq 0$$

EXAMPLE

Minimize $-2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$

subject to $-x_4^{1/2} + x_2^{1/3}x_3 \le -1$

$x_1, x_2, x_3, x_4 > 0$

$$
\begin{aligned}
-\delta_1 + \delta_2 + \delta_3 \qquad\qquad &= \zeta_0 \\
-\delta_4 + \delta_5 &= -\lambda_1
\end{aligned}
$$

EXAMPLE

Minimize $-2x_1x_2x_3^4x_4^{-1} + x_2^{-1}x_3^{-1} + 5x_1^{1/2}x_4$

subject to $-x_4^{1/2} + x_2^{1/3}x_3 \le -1$

$x_1, x_2, x_3, x_4 > 0$

$$
\begin{aligned}
-\delta_1 \qquad + \tfrac{1}{2}\delta_3 \qquad\qquad &= 0 \\
-\delta_1 + \delta_2 \qquad\qquad + \tfrac{1}{3}\delta_5 &= 0 \\
-4\delta_1 - \delta_2 \qquad\qquad + \delta_5 &= 0 \\
\delta_1 \qquad + \delta_3 - \tfrac{1}{2}\delta_4 \qquad &= 0
\end{aligned}
$$

Orthogonality Constraints

$$\text{Maximize } \zeta_0 \left[\left(\frac{2}{\delta_1}\right)^{-\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{5}{\delta_3}\right)^{\delta_3} \left(\frac{1}{\delta_4}\right)^{-\delta_4} \left(\frac{1}{\delta_5}\right)^{\delta_5} \lambda_1^{-\lambda_1} \right]^{\zeta_0}$$

subject to

$$\begin{aligned}
-\delta_1 + \delta_2 + \delta_3 &= \zeta_0 \\
-\delta_4 + \delta_5 &= -\lambda_1 \\
-\delta_1 + \tfrac{1}{2}\delta_3 &= 0 \\
-\delta_1 + \delta_2 + \tfrac{1}{3}\delta_5 &= 0 \\
-4\delta_1 - \delta_2 + \delta_5 &= 0 \\
\delta_1 + \delta_3 - \tfrac{1}{2}\delta_4 &= 0
\end{aligned}$$

$$\delta_i \geq 0,\ i=1,\cdots 5;\ \lambda_1 \geq 0$$

Pseudo-Dual Signomial GP Problem

terms = 5;
primal variables = 4

$\Rightarrow$ degree of difficulty = zero

Let's guess that $\zeta_0 = +1$, i.e., that the optimal primal objective is positive...
The unique solution of the dual constraints is

$$\delta_1^* = 2/3,\ \delta_2^* = 1/3,\ \delta_3^* = 4/3,\ \delta_4^* = 4,\ \delta_5^* = 3,\ \lambda_1^* = 1$$

Since $\delta_i \geq 0$ & $\lambda_1 \geq 0$, the guess $\zeta_0 = +1$ was correct.

$$v(\delta^*, \lambda^*) = \left(\frac{2}{2/3}\right)^{-2/3} \left(\frac{1}{1/3}\right)^{1/3} \left(\frac{5}{4/3}\right)^{4/3} \left(\frac{1}{4}\right)^{-4} \left(\frac{1}{3}\right)^{3} 1^{-1} = 38.322$$

Computing the Primal Optimum

$$2x_1x_2x_3^4x_4^{-1} = \delta_1^* g_0(x^*) = \tfrac{2}{3}\times(38.322) \Rightarrow x_1x_2x_3^4x_4^{-1} = 12.77$$

$$x_2^{-1}x_3^{-1} = \delta_2^* g_0(x^*) = \tfrac{1}{3}\times(38.322) \Rightarrow x_2^{-1}x_3^{-1} = 12.77$$

$$5x_1^{1/2}x_4 = \delta_3^* g_0(x^*) = \tfrac{4}{3}\times(38.322) \Rightarrow x_1^{1/2}x_4 = 10.22$$

$$x_4^{1/2} = \delta_4^*/\lambda_1^* = 4 \Rightarrow x_4^{1/2} = 4$$

$$x_2^{1/3}x_3 = \delta_5^*/\lambda_1^* = 3 \Rightarrow x_2^{1/3}x_3 = 3$$

5 nonlinear equations with 4 unknowns

Computing the Primal Optimum

Taking logarithms of each side yields equations which are linear in the logarithms of the primal variables:

$$\begin{cases} \ln x_1 + \ln x_2 + 4\ln x_3 - \ln x_4 = \ln 12.77 \\ -\ln x_2 - \ln x_3 = \ln 12.77 \\ \tfrac{1}{2}\ln x_1 + \ln x_4 = \ln 10.22 \\ \tfrac{1}{2}\ln x_4 = \ln 4 \\ \tfrac{1}{3}\ln x_2 + \ln x_3 = \ln 3 \end{cases}$$

Computing the Primal Optimum

$$\begin{cases} x_1^* = 0.4079 \\ x_2^* = 0.004216 \\ x_3^* = 18.57 \\ x_4^* = 16 \end{cases}$$

Example

Maximize $5x_1^2 - x_2^2 x_3^4$

subject to

$$\frac{-5x_1^2}{x_2^2} + \frac{3x_3}{x_2} \geq 2$$

$$x_1 > 0,\ x_2 > 0,\ x_3 > 0$$