Solving Signomial GP Problems by Condensation

Minimize x_1

subject to

$$(x_1 - 2)^2 + (x_2 - 4)^2 \geq 4$$

X is outside a circle centered at (2,4) with radius 2

$$(x_1 - 3)^2 + (x_2 - 3)^2 \leq 4$$

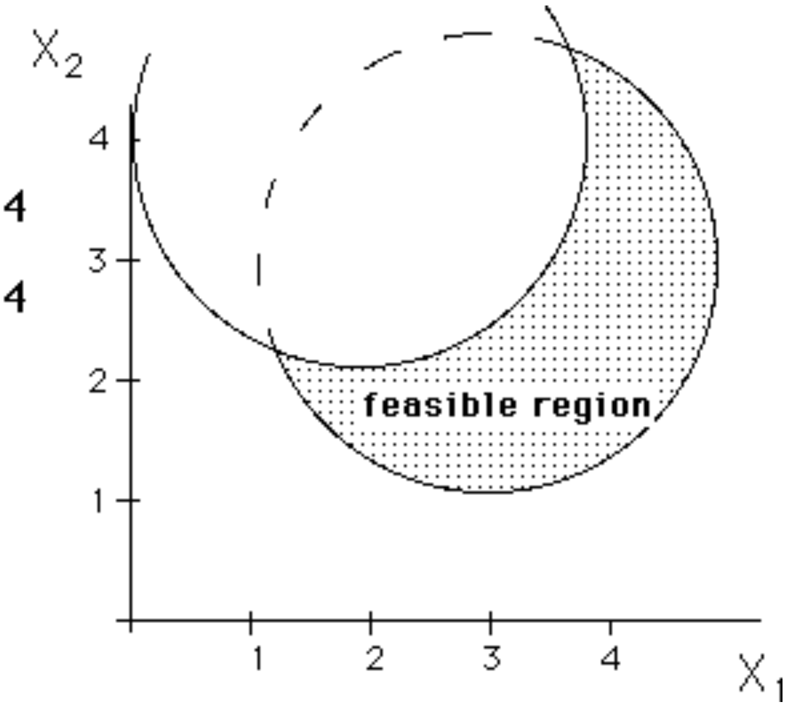
X is within a circle centered at (3,3) with radius 2

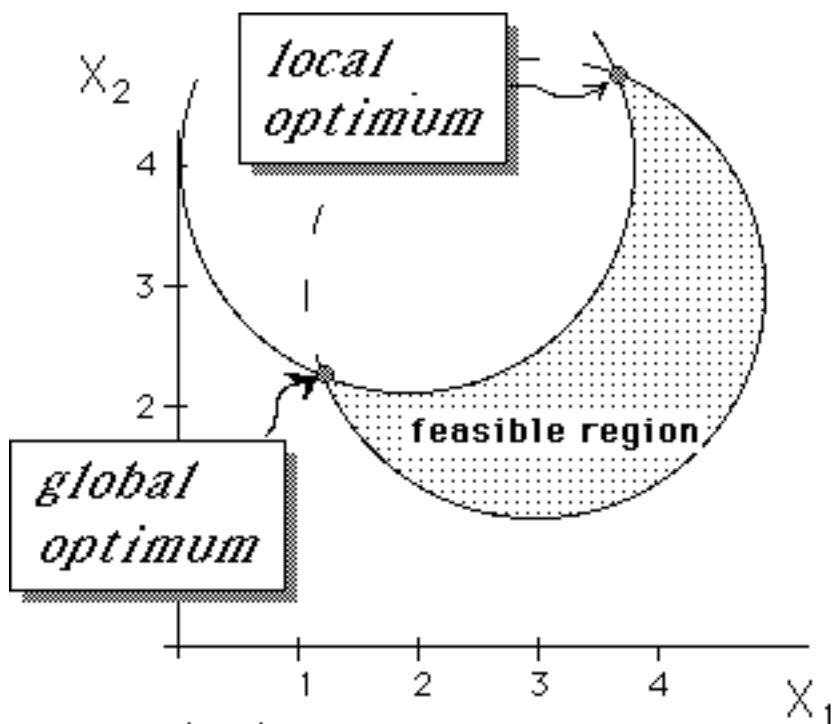
Minimize x_1

subject to

$$(x_1 - 2)^2 + (x_2 - 4)^2 \geq 4$$

$$(x_1 - 3)^2 + (x_2 - 3)^2 \leq 4$$





Reformulation as a GP problem

constraint # 1

$$\left(X_1 - 2\right)^2 + \left(X_2 - 4\right)^2 \geq 4$$

$$\Rightarrow \quad \left(x_1^2 - 4x_1 + 4\right) + \left(x_2^2 - 8x_2 + 16\right) \geq 4$$

$$\Rightarrow \quad - x_1^2 + 4x_1 - x_2^2 + 8x_1 \leq 16$$

The constraint becomes the signomial constraint

$$\Rightarrow \quad \boxed{\frac{X_1}{4} + \frac{X_2}{2} - \frac{X_1^2}{16} - \frac{X_2^2}{16} \leq 1}$$

Reformulation as a GP problem

constraint # 2

$$\left(X_1 - 3\right)^2 + \left(X_2 - 3\right)^2 \leq 4$$

$$\Rightarrow \quad \left(x_1^2 - 6x_1 + 9\right) + \left(x_2^2 - 6x_2 + 9\right) \leq 4$$

$$\Rightarrow \quad x_1^2 - 6x_1 + x_2^2 + 14 \leq 6x_2$$

The constraint becomes the signomial constraint

$$\Rightarrow \quad \boxed{\frac{X_1^2 X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} - X_1 X_2^{-1} \leq 1}$$

Signomial Geometric Program

Minimize X_1
subject to

$$\frac{X_1}{4} + \frac{X_2}{2} - \frac{X_1^2}{16} - \frac{X_2^2}{16} \leq 1$$

$$\frac{X_1^2 X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} - X_1 X_2^{-1} \leq 1$$

$$X_1 > 0,\ X_2 > 0$$

⇦ ⇨

To condense the signomial constraint

$$\frac{X_1}{4} + \frac{X_2}{2} - \frac{X_1^2}{16} - \frac{X_2^2}{16} \leq 1$$

we first write it in the form

$$\frac{X_1}{4} + \frac{X_2}{2} \leq 1 + \frac{X_1^2}{16} + \frac{X_2^2}{16}$$

$$\Longrightarrow \frac{\frac{X_1}{4} + \frac{X_2}{2}}{1 + \frac{X_1^2}{16} + \frac{X_2^2}{16}} \leq 1 \quad \Longrightarrow \quad \frac{0.25X_1 + 0.5X_2}{1 + 0.0625\ X_1^2 + 0.0625\ X_2^2} \leq 1$$

We next condense the denominator of

$$\frac{0.25X_1 + 0.5X_2}{1 + 0.0625\ X_1^2 + 0.0625\ X_2^2} \le 1$$

into a single term. Let's use the point Xo = (4,5) at which the terms of the denominator are

1 + 1 + 1.5626 = 3.5625

Then

$$\delta_1 = \delta_2 = \frac{1}{3.5625} = 0.2807 \quad \text{and} \quad \delta_3 = \frac{1.5625}{3.5625} = 0.4386$$

$$\delta_1 = \delta_2 = 0.2807, \quad \delta_3 = 0.4386$$

Coefficient:

$$C(\delta) = \prod_{i=1}^{3} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}$$

$$C(\delta) = \left(\frac{1}{0.2807}\right)^{0.2807} \left(\frac{0.0625}{0.2807}\right)^{0.2807} \left(\frac{0.0625}{0.4386}\right)^{0.4386}$$

= 0.3987

$$\delta_1 = \delta_2 = 0.2807, \quad \delta_3 = 0.4386$$

Exponents:

$$\mathbf{a}_j(\delta) = \sum_{i=1}^{3} \mathbf{a}_{ij}\delta_i$$

$$\mathbf{a}_1 = 0\delta_1 + 2\delta_2 + 0\delta_3 = 2(0.2807) = 0.5614$$

$$\mathbf{a}_2 = 0\delta_1 + 0\delta_2 + 2\delta_3 = 2(0.4386) = 0.8772$$

$C(\delta) = 0.3987$

$\mathbf{a}_1 = 0.5614$

$\mathbf{a}_2 = 0.8772$

Condensed denominator is

$$0.3987\, X_1^{0.5614}\, X_2^{0.8772}$$

Geometric Inequality implies

$$1 + 0.0625X_1^2 + 0.0625\ X_2^2 \ \geq \ 0.3987\ X_1^{0.5614}\ X_2^{0.8772}$$

and so

$$\frac{0.25X_1 + 0.5X_2}{1 + 0.0625\ X_1^2 + 0.0625\ X_2^2} \leq \frac{0.25X_1 + 0.5X_2}{0.3987\ X_1^{0.5614}\ X_2^{0.8772}}$$

$$\frac{0.25X_1 + 0.5X_2}{0.3987\ X_1^{0.5614}\ X_2^{0.8772}}$$

$$= \frac{0.25}{0.3987} X_1^{1-0.5614}\ X_2^{-0.8772} + \frac{0.5}{0.3987} X_1^{-0.5614}\ X_2^{1-0.8772}$$

$$= 0.627\ X_1^{0.4386}\ X_2^{-0.8772} + 1.254\ X_1^{-0.5614}\ X_2^{0.1228}$$

which is a posynomial!

If we constrain this posynomial so as to be ≤ 1, then by the geometric inequality, the original signomial should also be ≤ 1.

That is, any X feasible in the posynomial constraint derived by condensation will also be feasible in the signomial constraint:

$$\frac{0.25X_1 + 0.5X_2}{1 + 0.0625\ X_1^2 + 0.0625\ X_2^2}$$

$$\leq\ 0.627\ X_1^{0.4386}\ X_2^{-0.8772} + 1.254\ X_1^{-0.5614}\ X_2^{0.1228}\ \leq\ 1$$

The second signomial constraint may be condensed in a similar fashion:

$$\frac{X_1^2X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} - X_1X_2^{-1} \leq 1$$

$$\Longrightarrow \quad \frac{X_1^2X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} \leq 1 + X_1X_2^{-1}$$

$$\Longrightarrow \quad \frac{\frac{X_1^2X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3}}{1 + X_1X_2^{-1}} \leq 1$$

$$\frac{\frac{X_1^2X_2^{-1}}{6}+\frac{X_2}{6}+\frac{7X_2^{-1}}{3}}{1+X_1X_2^{-1}} \le 1$$

At (4,5), the denominator is 1 + 0.8 = 1.8, so

$$\delta_1 = \frac{1}{1.8} = 0.555,\ \delta_2 = \frac{0.8}{1.8} = 0.444$$

can be condensed (using $\delta_1 = 0.555,\ \delta_2 = 0.444$) into the posynomial constraint

$$0.08385X_1^{1.555}X_2^{-0.555} + 0.08385X_1^{-0.444}X_2^{1.444} + 1.174X_1^{-0.444}X_2^{-0.555} \le 1$$

The signomial GP problem is therefore approximated by the posynomial problem:

Minimize X_1

subject to

$$0.627\, X_1^{0.4386}\, X_2^{-0.8772} + 1.254\, X_1^{-0.5614}\, X_2^{0.1228} \le 1$$

$$0.08385X_1^{1.555}X_2^{-0.555} + 0.08385X_1^{-0.444}X_2^{1.444} + 1.174X_1^{-0.444}X_2^{-0.555} \le 1$$

$$X_1 > 0,\ X_2 > 0$$