Examples: Unconstrained Optimization

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

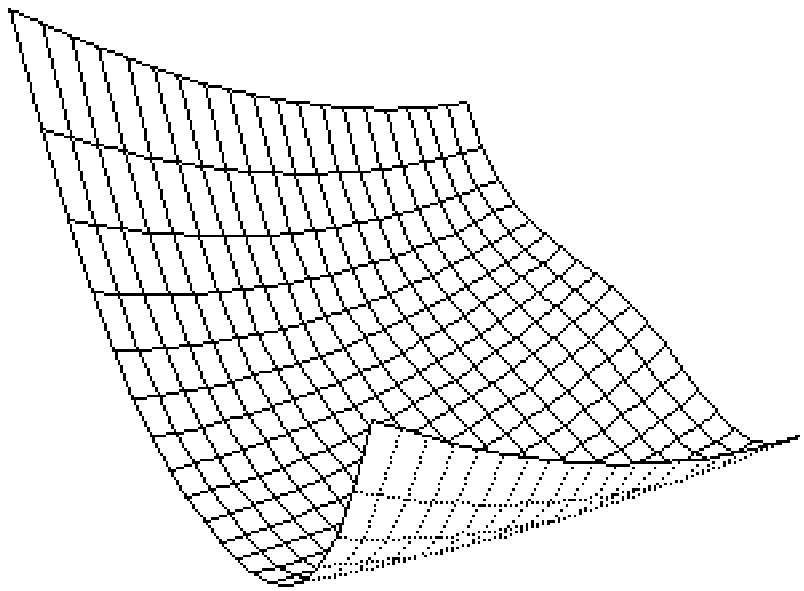
author

$$f(x_1,x_2) = (x_2 - x_1^2)^2 + (1 - x_1)^2$$

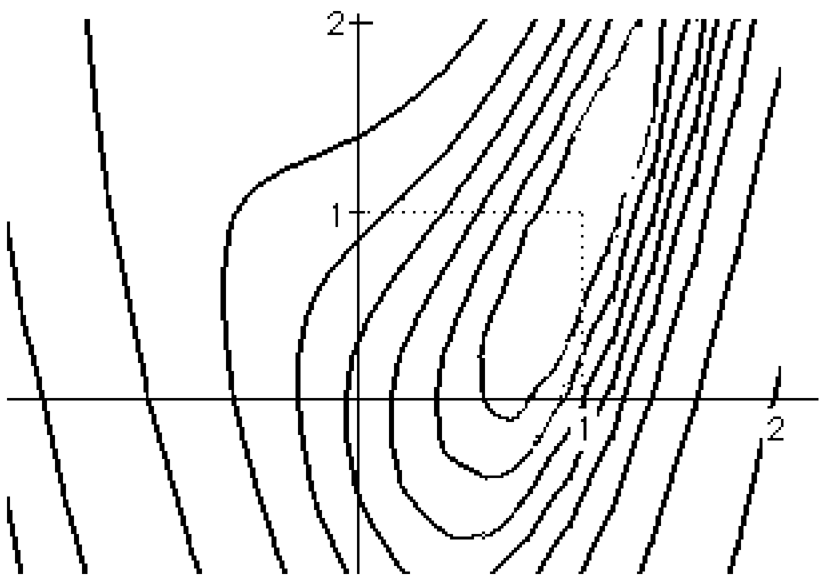
```
     ∇Z←F X
[1]   ⍝
[2]   ⍝         TEST FUNCTION FOR UNCONSTRAINED MINIMIZATION
[3]   ⍝
[4]   f_evaluations←f_evaluations+1
[5]    Z←((X[2]-X[1]*2)*2)+(1-X[1])*2
     ∇
```

$$\nabla f(x_1,x_2) = \begin{bmatrix} 4x_1^3 - 4x_1x_2 + 2x_1 - 2 \\ 2\,(x_2 - x_1^2) \end{bmatrix}$$

```
     ∇G←GRADIENT X
[1]   ⍝
[2]   ⍝       GRADIENT OF TEST FUNCTION FOR
[3]   ⍝         UNCONSTRAINED MINIMIZATION
[4]   ⍝
[5]   gradient_evaluations←gradient_evaluations+1
[6]    G←(4×X[1]*3)+(¯4×X[1]×X[2])+(2×X[1])-2
[7]    G←G,2×(X[2]-X[1]*2)
     ∇
```



Contours of $f(x_1,x_2) = (x_2 - x_1^2)^2 + (1 - x_1)^2$



$f(x_1,x_2) \geq 0$,
& $f(1,1) = 0$
$\Rightarrow x^* = (1,1)$

Tolerances for Search

Stopping Criteria:

One-dimensional search (max |derivative|):	0.001
One-dimensional search (|Δx|):	0.01
Maximum Abs. Value of partial derivatives:	0.001
Improvement in objective function:	0.00001
Length of step:	0.001
Simplex: stopping criterion:	0.0001
alpha	1.1
beta	0.5
gamma	1.5

Maximum number of iterations

25 for 1-dimensional search
50 for search algorithm

Numerical Example

- ☞ Newton's Method
- ☞ Newton's Method with Linesearch
- ☞ Steepest Descent Method
- ☞ Fletcher-Reeves Conjugate Gradient Method
- ☞ Davidon-Fletcher-Powell (DFP) Method
- ☞ Powell's Method

Newton's Algorithm

Minimize $f(x_1,x_2) = (x_2 - x_1^2)^2 + (1 - x_1)^2$

The optimum is at $x^* = (1,1)$

We will use $x^\circ = (-2,+2)$

Newton's Algorithm

Iteration 1

x = ⁻2 2
F(x) = 13

∇F(x) = ⁻22 ⁻4

Hessian Matrix = $\begin{bmatrix} 12 & 8 \\ 8 & 2 \end{bmatrix}$

Step: Δx= 0.6 ⁻0.4,
with magnitude 0.7211102551 ← $\begin{bmatrix} 12 & 8 \\ 8 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -22 \\ -4 \end{bmatrix}$

Iteration 2

x = ¯1.4 1.6 ← $x^0 + \Delta x = (-2, 2) + (0.6, -0.4)$
F(x) = 5.8896

Improvement: 7.1104

∇F(x) = ¯6.816 ¯0.72

Hessian Matrix = $\begin{bmatrix} 19.12 & 5.6 \\ 5.6 & 2 \end{bmatrix}$

Step: Δx= 1.395348837 ¯3.546976744,
with magnitude 3.811566922

Iteration 3

x = ¯0.004651162791 ¯1.946976744
F(x) = 4.800126641

Improvement: 1.089473359

∇F(x) = ¯2.045525551 ¯3.893996755

Hessian Matrix = $\begin{bmatrix} 9.788166577 & 0.01860465116 \\ 0.01860465116 & 2 \end{bmatrix}$

Step: Δx= 0.2052823516 1.945088774,
with magnitude 1.955891404

Iteration 4

x = 0.2006311889 ¯0.001887969956
F(x) = 0.640766347

Improvement: 4.159360294

∇F(x) = ¯1.564918552 ¯0.08428168779

Hessian Matrix = $\begin{bmatrix} 2.490586367 & -0.8025247554 \\ -0.8025247554 & 2 \end{bmatrix}$

Step: Δx= 0.7372335253 0.3379649212,
with magnitude 0.8110077428

Iteration 5

x = 0.9378647142 0.3360769512
F(x) = 0.2992674694

Improvement: 0.3414988776

∇F(x) = 1.914697102 ¯1.087026542

Hessian Matrix = $\begin{bmatrix} 11.21077486 & -3.751458857 \\ -3.751458857 & 2 \end{bmatrix}$

Step: Δx= 0.02977215889 0.5993577855,
with magnitude 0.6000967726

Iteration 6

x = 0.9676368731 0.9354347367
F(x) = 0.001048157656

Improvement: 0.2982193117

∇F(x) = ¯0.06129547236 ¯0.00177276289

Hessian Matrix = $\begin{bmatrix} 9.494114471 & ^-3.870547492 \\ ^-3.870547492 & 2 \end{bmatrix}$

Step: Δx= 0.0323058563 0.06340705698,
with magnitude 0.07116265331

Iteration 7

x = 0.9999427294 0.9988417937
F(x) = 1.092523551E¯6

Improvement: 0.001047065133

∇F(x) = 0.004059893072 ¯0.002087336702

Hessian Matrix = $\begin{bmatrix} 10.00325837 & ^-3.999770918 \\ ^-3.999770918 & 2 \end{bmatrix}$

Step: Δx= 0.00005715132911 0.001157964463,
with magnitude 0.001159373957

Iteration 8

```
x = 0.9999998807 0.9999997581
F(x) = 1.424174293E¯14

Improvement: 1.092523537E¯6

∇F(x) = ¯2.25523038E¯7 ¯6.532548724E¯9
```

Hessian Matrix = $\begin{bmatrix} 9.999998104 & ¯3.999999523 \\ ¯3.999999523 & 2 \end{bmatrix}$

Convergence criterion satisfied

```
Improvement in objective
Gradient
*** CONVERGED ***
```



```
Solution found is 0.9999998807 0.9999997581
where F is 1.424174293E¯14
and ∇F is ¯2.25523038E¯7 ¯6.532548724E¯9

# iterations = 8
# function evaluations= 9
# gradient evaluations= 9
# hessian evaluations= 8
Elapsed CPU time: 23.95 seconds
```

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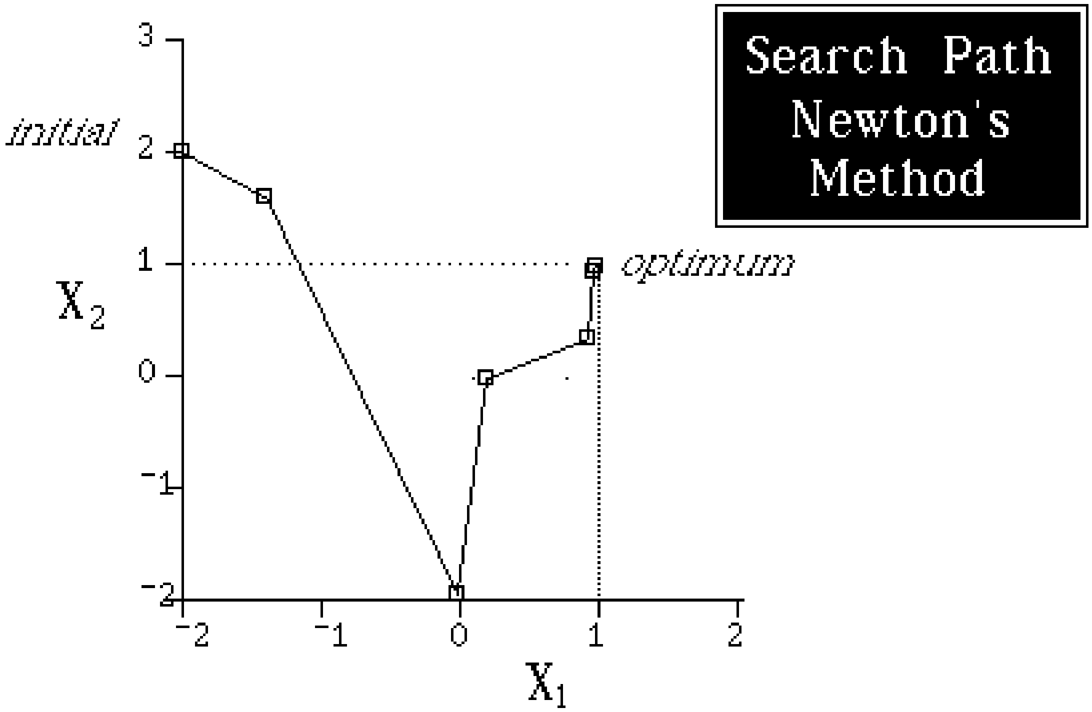
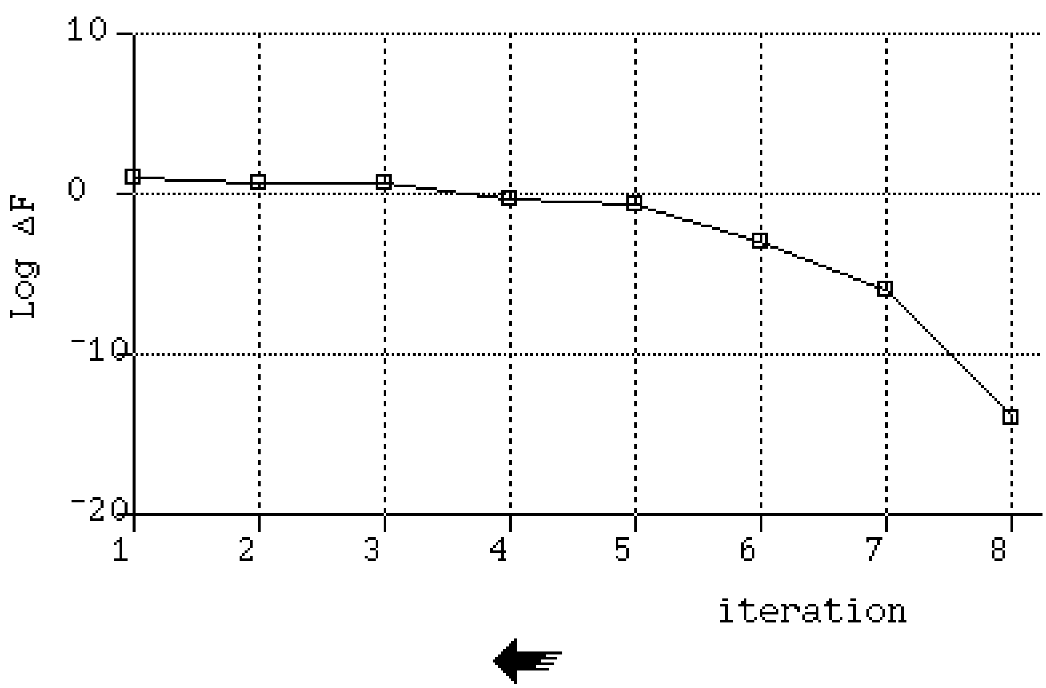

Search Path Newton's Method
initial
optimum
3
2
1
0
-1
-2
X_2
-2
-1
0
1
2
X_1


10
0
-10
-20
Log ΔF
1
2
3
4
5
6
7
8
iteration

Newton's Method with Linesearch

Newton's Method

(incorporating 1-dimensional search)

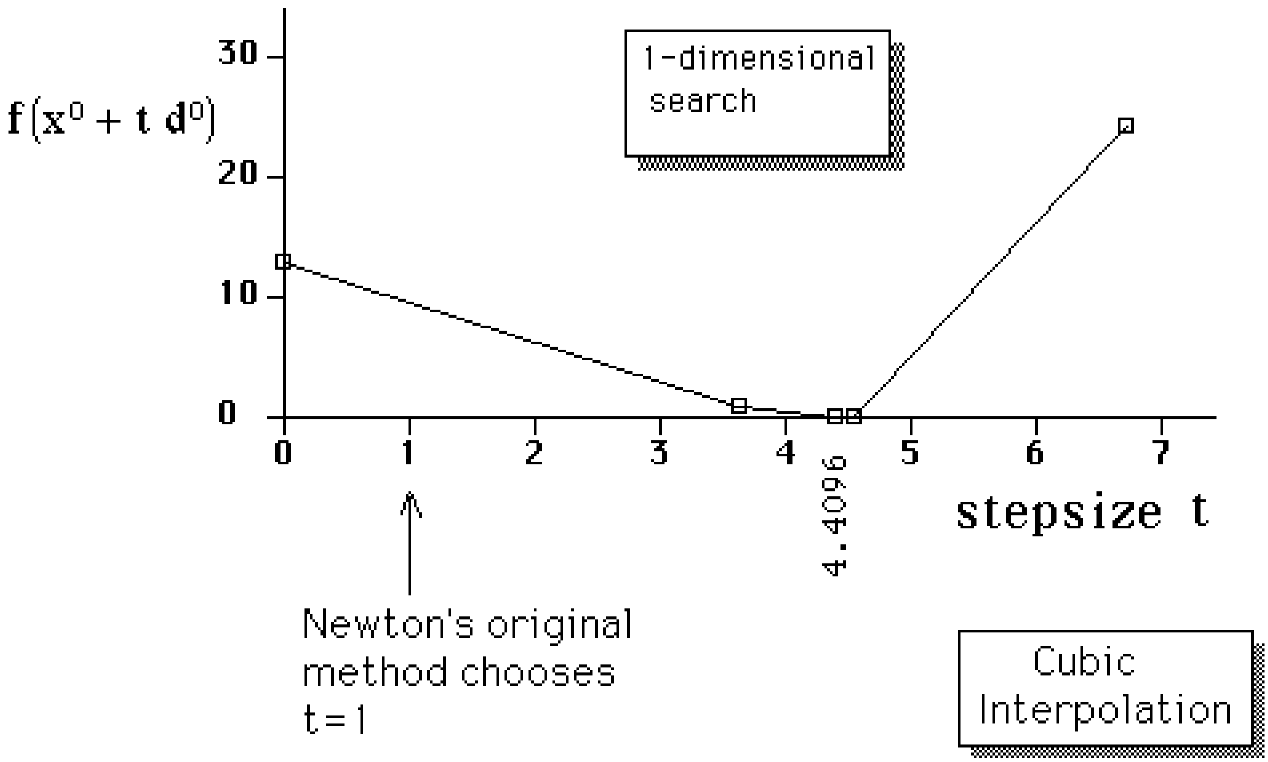
Iteration 1

```
x = ¯2 2
F(x) = 13

∇F(x) = ¯22 ¯4
```

$$\text{Hessian Matrix} = \begin{bmatrix} 42 & 8 \\ 8 & 2 \end{bmatrix}$$

```
Search direction is 0.6 ¯0.4
Optimal stepsize is 4.40964343
Step: Δx= 2.645786058 ¯1.763857372,
  with magnitude 3.179839099
```

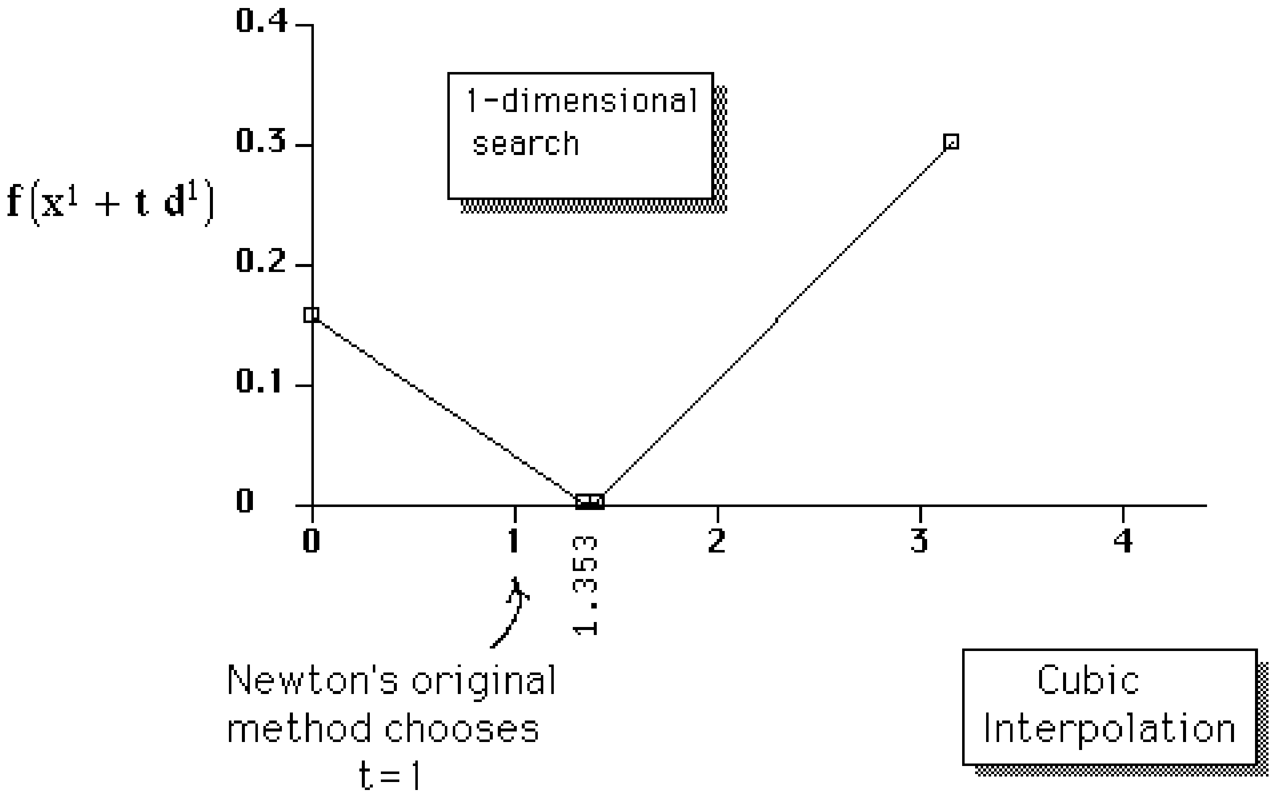
Much greater than that chosen by Newton's method (t=1)



Iteration 2

```
x = 0.6457860579 0.236142628
F(x) = 0.158191243

Improvement: 12.84180876

∇F(x) = ¯0.2411448302 ¯0.3617940092

Hessian Matrix = [  6.059905079 ¯2.583144232 ]
                 [ ¯2.583144232  2           ]

Search direction is 0.2601083128 0.5168456485

Optimal stepsize is 1.353002541

Step: Δx= 0.3519272081 0.6992934756,
  with magnitude 0.7828563884
```

← As we get nearer to the optimum, the optimal step size gets nearer to 1.0



Iteration 3

x = 0.997713266 0.9354361037
F(x) = 0.003604708074

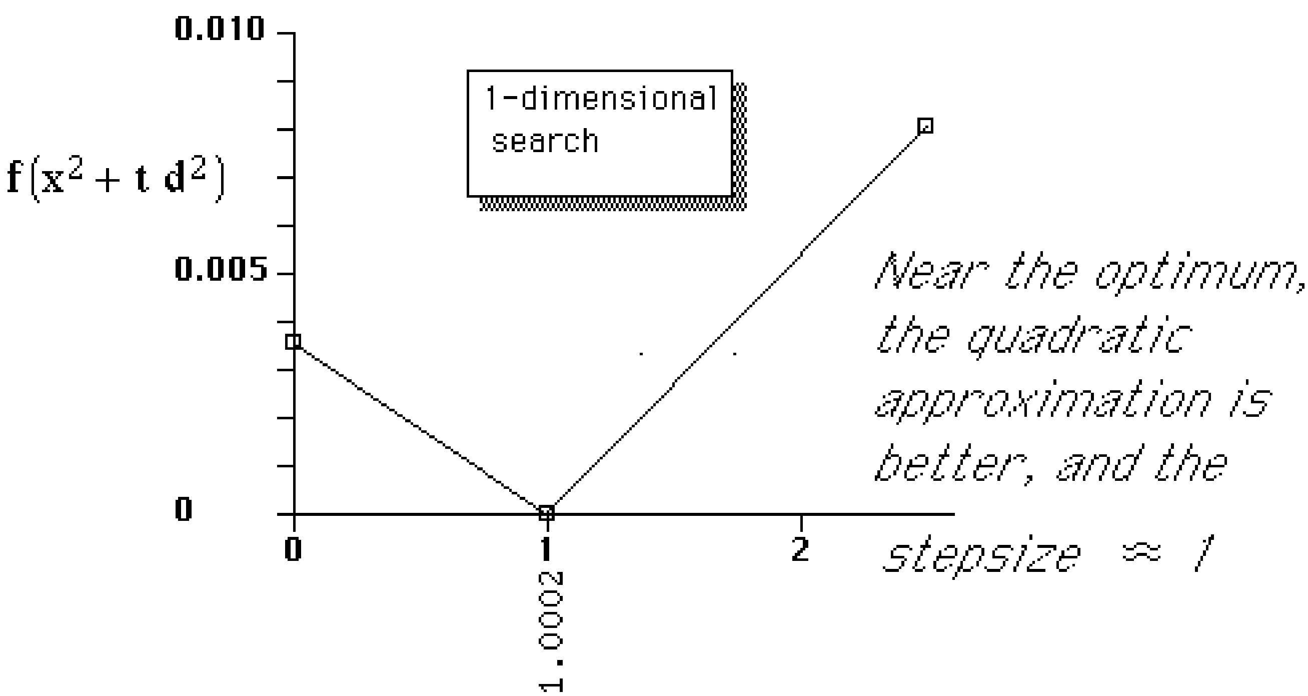
Improvement: 0.1545865349

∇F(x) = 0.2348603857 ¯0.119991315

Hessian Matrix = $\begin{bmatrix} 10.20343672 & ^{-}3.990853064 \\ ^{-}3.990853064 & 2 \end{bmatrix}$

Search direction is 0.002041742593 0.06406980486
Optimal stepsize is 1.000208261 ← $t \approx 1$

Step: Δx= 0.002042167809 0.06408314813,
with magnitude 0.06411567923



Iteration 4

x = 0.9997554338 0.9995192518
F(x) = 5.988190232E¯8

Improvement: 0.003604648192

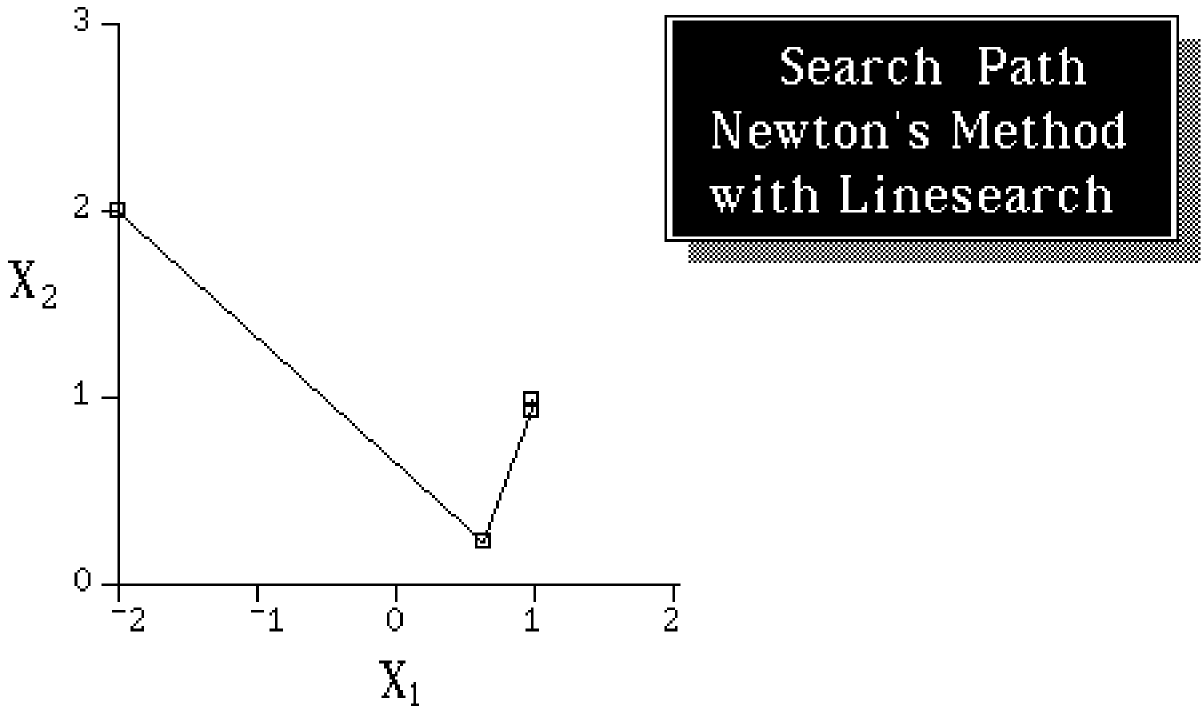
∇F(x) = ¯0.0005224214986 0.00001664865847

$$\text{Hessian Matrix} = \begin{bmatrix} 9.996054123 & ^{-}3.999021735 \\ ^{-}3.999021735 & 2 \end{bmatrix}$$

Convergence criterion satisfied:

Gradient

```
Convergence criterion satisfied:
Gradient

                *** CONVERGED ***

Solution found is 0.9997554338 0.9995192518
where F is 5.988190232E¯8
and ∇F is ¯0.0005224214986 0.00001664865847

# iterations = 4
# function evaluations= 37
# gradient evaluations= 21
# hessian evaluations= 4
Elapsed CPU time: 26.65 seconds
```

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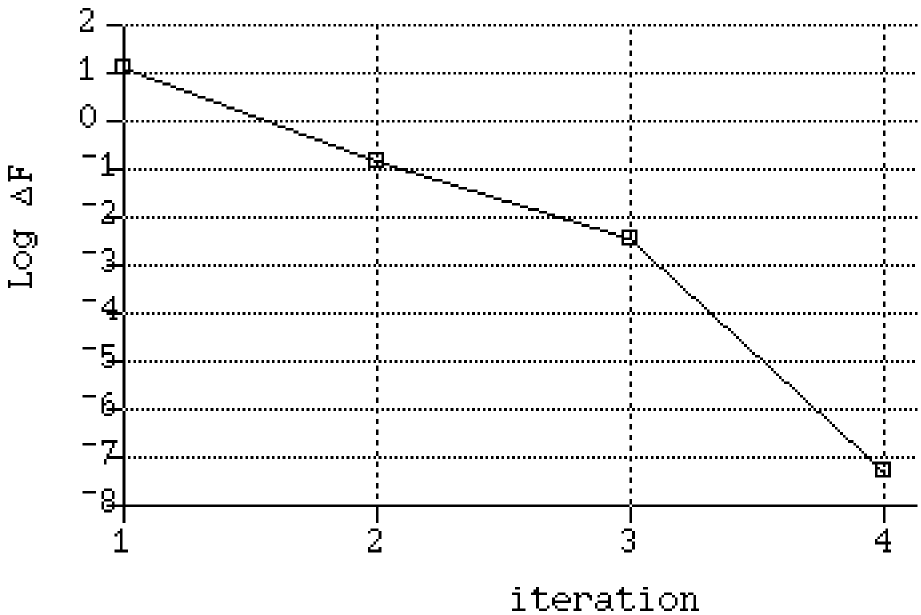

Steepest Descent

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Steepest Descent Algorithm

Iteration 1

$x = ^{-}2\ 2$
$F(x) = 13$
$\nabla F(x) = ^{-}22\ ^{-}4$

Search direction is $-\nabla F$ = 22 4

Optimal stepsize is 0.1621737068

Step is Δx= 3.567821549 0.6486948271,
with magnitude 3.626314325

Iteration 2

x = 1.567821549 2.648694827
$F(x)$ = 0.3587612676

Improvement: 12.64123873
$\nabla F(x) = ^{-}0.05985480199\ 0.3812608333$

Search direction is $-\nabla F$ = 0.05985480199 ¯0.3812608333

Optimal stepsize is 0.2275651301

Step is Δx= 0.0136208658 ¯0.08676167111,
with magnitude 0.08782434491

$$\begin{bmatrix} 22 \\ 4 \end{bmatrix}^T \begin{bmatrix} ^{-}0.05985480199 \\ 0.3812608333 \end{bmatrix} = -0.208238$$

Iteration 3

```
x = 1.581442415 2.561933156
F(x) = 0.3417929942

Improvement: 0.01696827346
∇F(x) = 0.7771834003 0.1219460874

Search direction is -∇F = ¯0.7771834003 ¯0.1219460874

Optimal stepsize is 0.05369486991

Step is Δx= ¯0.04173076157 ¯0.006547879298,
  with magnitude 0.04224134449
```

$$\begin{bmatrix} -0.05985480199 \\ 0.3812608333 \end{bmatrix}^T \begin{bmatrix} 0.7771834003 \\ 0.1219460874 \end{bmatrix} = 0.000025$$

Iteration 4

```
x = 1.539711654 2.555385277
F(x) = 0.325392897

Improvement: 0.0164000972
∇F(x) = ¯0.05795122546 0.3693466013

Search direction is -∇F = 0.05795122546 ¯0.3693466013

Optimal stepsize is 0.2303221653

Step is Δx= 0.01334745173 ¯0.08506870896,
  with magnitude 0.08610946354
```

.... *etc.*

Steepest Descent

After fifty iterations, the current solution is still relatively far from optimal!

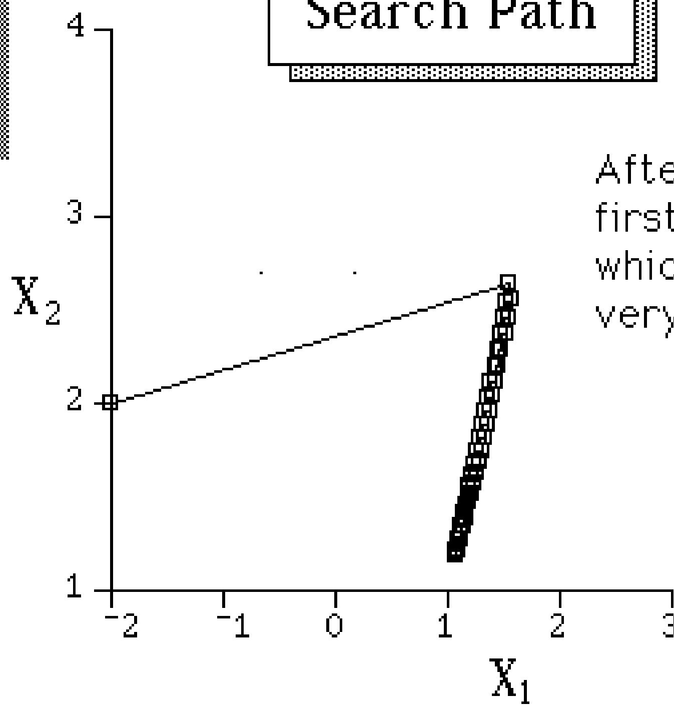
```
*** Warning: did not converge in 50 steps

Solution found is 1.080545748 1.178896815
where F is 0.006615707827
and ∇F is 0.1121743152 0.02263540436

# iterations = 51
# function evaluations= 320
# gradient evaluations= 263
Elapsed CPU time: 125.25 seconds
```


Steepest Descent

Search Path



After a rather large first step, the steps which follow are very small!

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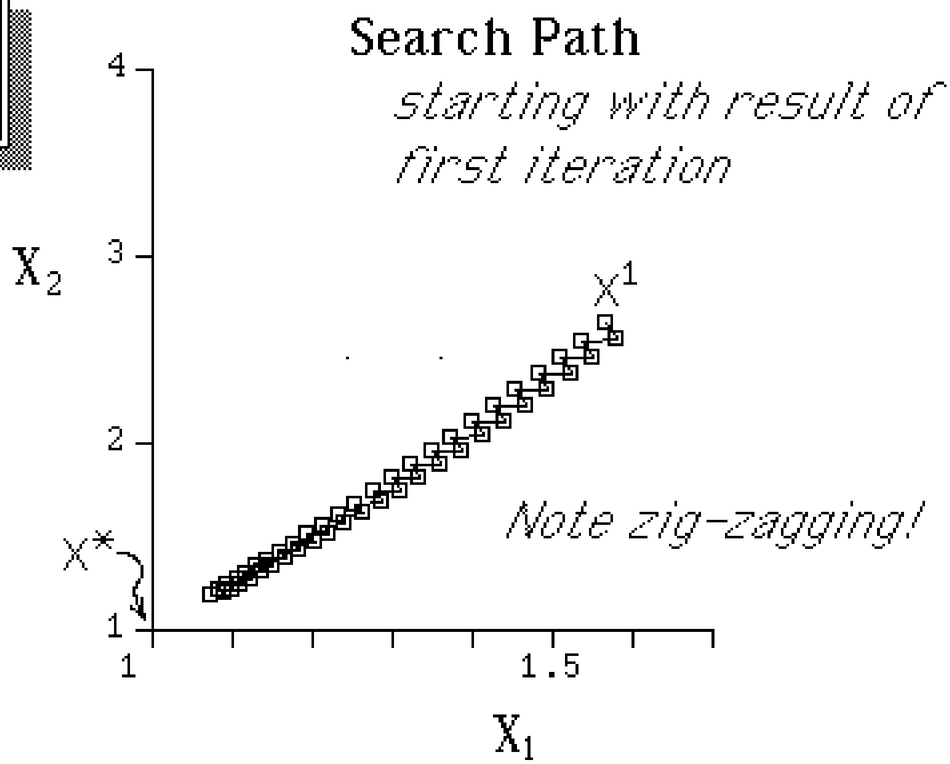
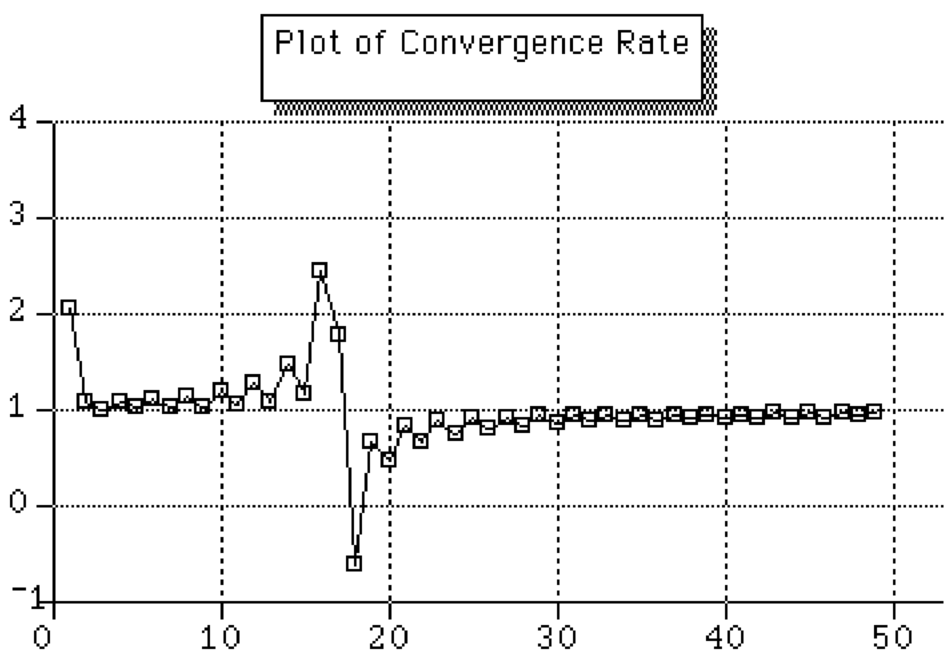
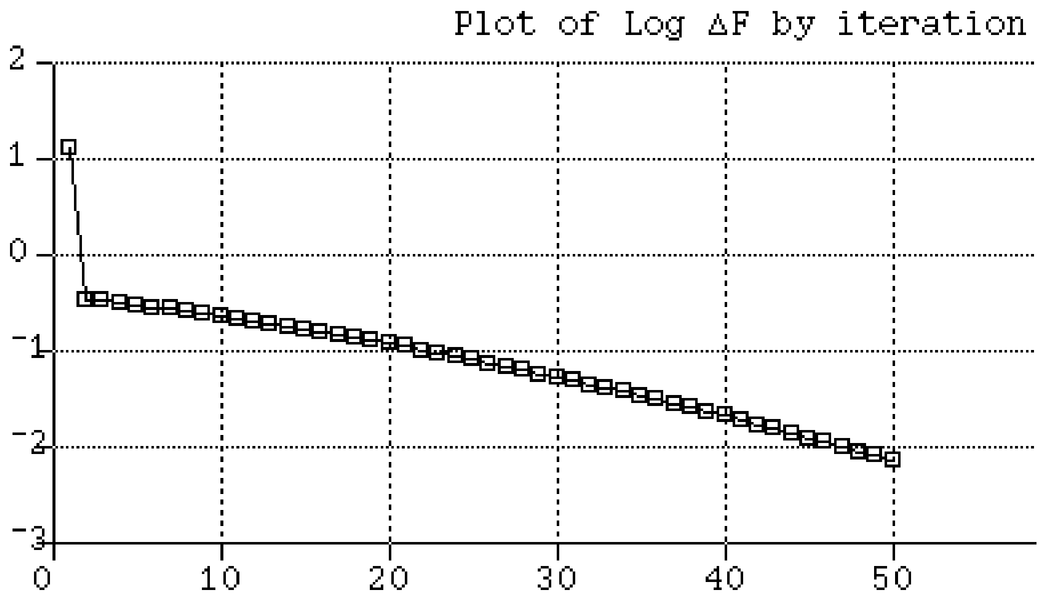
Steepest Descent







Fletcher-Reeves Algorithm

(Also known as "Conjugate Gradient" method)

The step direction is a combination of the steepest descent direction and the previous search direction:

$$d^k = -\nabla f(x^k) + \frac{\left\|\nabla f(x^k)\right\|^2}{\left\|\nabla f(x^{k-1})\right\|^2} d^{k-1}$$

Fletcher-Reeves Conjugate-Gradient Algorithm

Iteration 1

```
x = ¯2 2
F(x) = 13
∇F(x) = ¯22 ¯4
```

$\|\nabla f(x^0)\|^2 = 500$

New search direction is 22 4

initially, d is steepest descent direction

Optimal stepsize is 0.1621737068

Step: Δx= 3.567821549 0.6486948271,
  with magnitude 3.626314325

Iteration 2

```
x = 1.567821549 2.648694827
F(x) = 0.3587612676
```

Improvement is 12.64123873

∇F(x) = ¯0.05985480199 0.3812608333

$\|\nabla f(x^1)\|^2 = 0.14894$

$\frac{0.14894}{500}$

Multiplier for old search direction = 0.0002978848406

New search direction is 0.06640826848 ¯0.3800692939
  (= (-∇F) + 0.0002978848406 × (old search direction))

Optimal stepsize is 0.2124445147

Step: Δx= 0.01410807237 ¯0.08074363668,
  with magnitude 0.08196689924

Iteration 3

x = 1.581929622 2.56795119

F(x) = 0.3429257691
Improvement is 0.01583549852

$\|\nabla f(x^1)\|^2 = 0.5792$

∇F(x) = 0.7497109385 0.130899725

$\frac{0.5792}{0.14894}$

Multiplier for old search direction = 3.888759348

New search direction is ¯0.4914651637 ¯1.608897745
(= (-∇F) + 3.888759348 × (old search direction))

Optimal stepsize is 0.8659182831

Step: Δx= ¯0.4255686707 ¯1.393173973,
with magnitude 1.456723176

Iteration 4

x = 1.156360951 1.174777218

F(x) = 0.05082037349
Improvement is 0.2921053956

∇F(x) = 1.063863592 ¯0.3247868624

Multiplier for old search direction = 2.136204459

New search direction is ¯2.113733666 ¯3.112147673
(= (-∇F) + 2.136204459 × (old search direction))

Optimal stepsize is 0.0841505677

Step: Δx= ¯0.1778718879 ¯0.2618889935,
with magnitude 0.3165821432

Iteration 5

x = 0.978489063 0.9128882243

F(x) = 0.00244765655
Improvement is 0.04837271694

∇F(x) = 0.1313551401 ¯0.08910524431

Multiplier for old search direction = 0.02036213953

New search direction is ¯0.17439528 0.02573525915
(= (-∇F) + 0.02036213953 × (old search direction))

Optimal stepsize is 0.07623555685

Step: Δx= ¯0.01329512128 0.001961941812,
with magnitude 0.01343910211

Iteration 6

x = 0.9651939418 0.9148501662

F(x) = 0.001491996689
Improvement is 0.0009556598605

∇F(x) = ¯0.004947291822 ¯0.03349835813

Multiplier for old search direction = 0.04551160803

New search direction is ¯0.002989717803 0.03466961116
(= (-∇F) + 0.04551160803 × (old search direction))

Optimal stepsize is 0.3486481657

Step: Δx= ¯0.001042359628 0.01208749634,
with magnitude 0.01213235679

Iteration 7

x = 0.9641515821 0.9269376625

F(x) = 0.001292134801
Improvement is 0.0001998618881

∇F(x) = ¯0.06147447316 ¯0.005301221693

Multiplier for old search direction = 3.320392197

New search direction is 0.0515474375 0.120417928
(= (-∇F) + 3.320392197 × (old search direction))

Optimal stepsize is 0.6319461373

Step: Δx= 0¯.03257520401 0.07609764448,
with magnitude 0.08277678064

Iteration 8

x = 0.9967267862 1.003035307

F(x) = 0.0001023183669
Improvement is 0.001189816434

∇F(x) = ¯0.04470519867 0.01914204148

Multiplier for old search direction = 0.6211819624

New search direction is 0.07672553705 0.05565940336
(= (-∇F) + 0.6211819624 × (old search direction))

Optimal stepsize is 0.07691320942

Step: Δx= 0.005901207299 0.004280943346,
with magnitude 0.007290454274

Iteration 9

x = 1.002627993 1.00731625

F(x) = 0.00001112262483
Improvement is 0.00009119574211

∇F(x) = ¯0.002979026213 0.004106714139

Multiplier for old search direction = 0.01088371968

New search direction is 0.003814085451 ¯0.003500932795
(= (-∇F) + 0.01088371968 × (old search direction))

Optimal stepsize is 0.09269772415

Step: Δx= 0.000353557041 ¯0.0003245285025,
with magnitude 0.0004799180453

Iteration 10

x = 1.00298155 1.006991722

F(x) = 9.929495032E¯6
Improvement is 1.193129795E¯6

∇F(x) = 0.001872014698 0.002039462382

Multiplier for old search direction = 0.2977441996

<u>Convergence criterion satisfied:</u>

Improvement in objective
Step size

Convergence criterion satisfied:

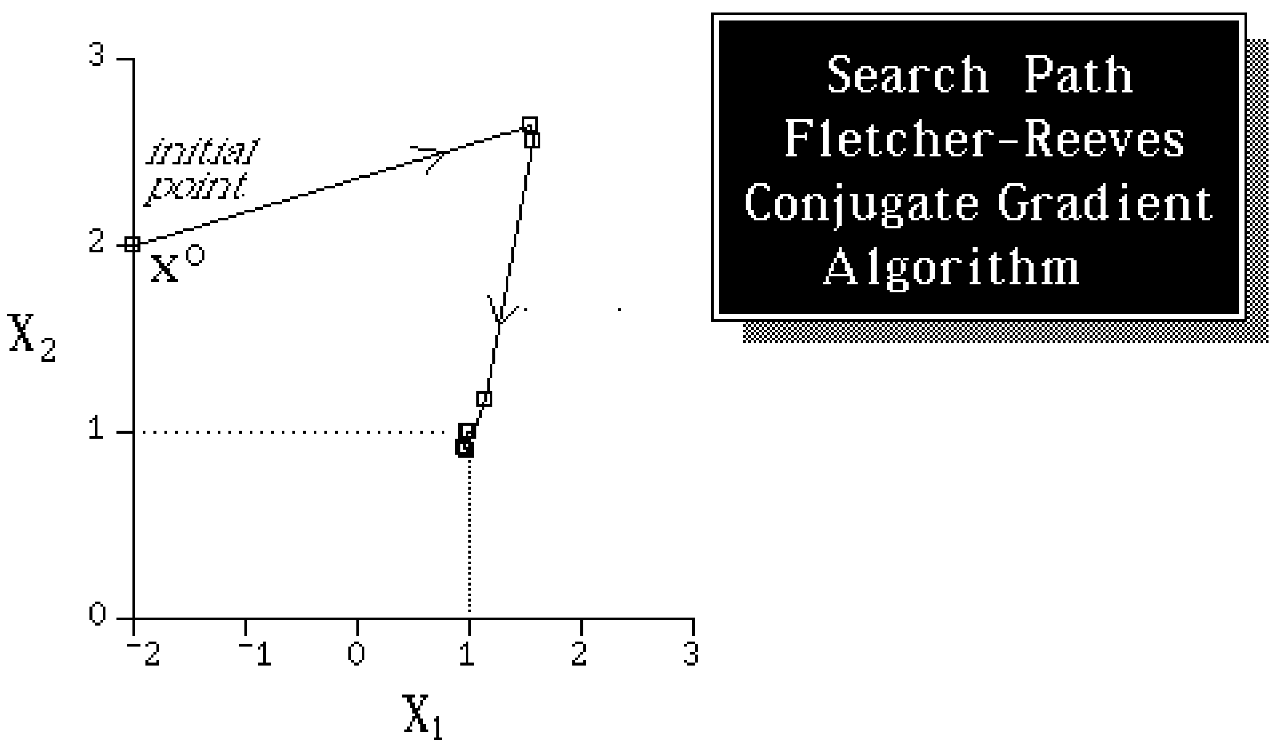
Improvement in objective
Step size

*** CONVERGED ***

Solution found is 1.00298155 1.006991722

where F is 9.929495032E⁻6
and ∇F is 0.001872014698 0.002039462382

\# iterations = 10
\# function evaluations= 80
\# gradient evaluations= 51
Elapsed CPU time: 52.65 seconds



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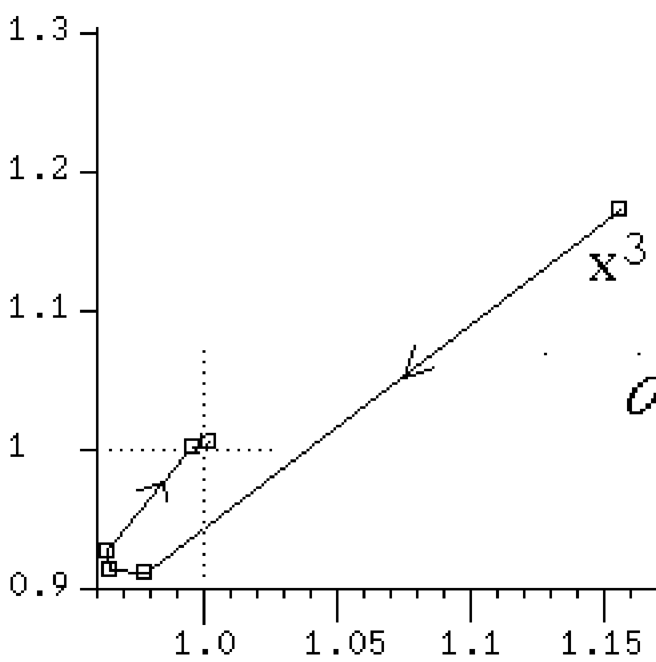
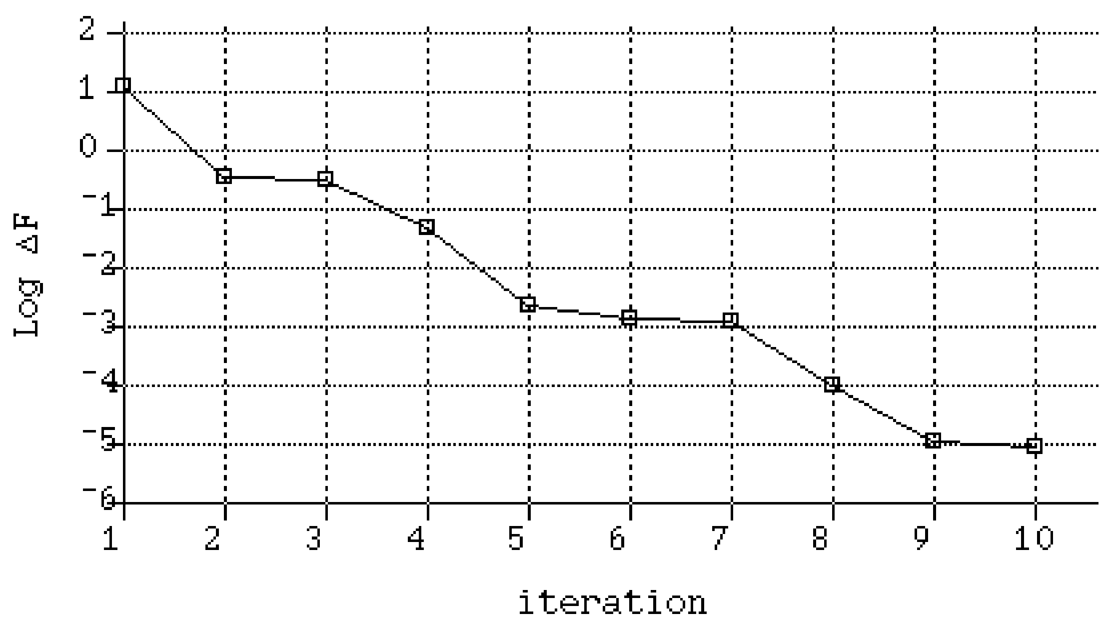
Search Path Fletcher-Reeves Conjugate Gradient Algorithm



Omitting x^0, x^1, & x^2 from the plot



Davidon-Fletcher-Powell Algorithm

Iteration 1

```
x = ¯2 2
F(x) = 13
∇F(x) = ¯22 ¯4

Q (approx. of Hessian inverse)  Actual Hessian inverse

          ┌1 0┐                         ┌ 0.1 ¯0.4┐
          └0 1┘                         └¯0.4  2.1┘

Search direction is 22 4
Optimal stepsize is 0.1621737068
Step is Δx= 3.567821549 0.6486948271,
  with magnitude 3.626314325
Δgradient = 21.9401452 4.381260833

Q updated: Q ← Q + A + B, where matrices A,B =

  ┌0.156918  0.028530┐        ┌¯0.961652  ¯0.192033┐
  └0.028530  0.005187┘        └¯0.192033  ¯0.038347┘
```

Iteration 2

x = 1.567821549 2.648694827
F(x) = 0.3587612676
Improvement is 12.64123873
∇F(x) = ¯0.05985480199 0.3812608333

Q (approx. of Hessian inverse)

0.1952663236	¯0.1635031489
¯0.1635031489	0.96683987

Actual Hessian inverse

0.8080949565	2.533897373
2.533897373	8.445397811

Search direction is 0.07402497392 ¯0.3784046231
Optimal stepsize is 0.196711875
Step is Δx= 0.01456159142 ¯0.07443668292,
with magnitude 0.07584760846
Δgradient = 0.7795173394 ¯0.240617353

Q updated: Q ← Q + A + B, where matrices A,B =

0.00724631285	¯0.0370420702
¯0.03704207027	0.189353537

¯0.155503537	0.2923208779
0.292320877	¯0.5495148012

Iteration 3

x = 1.582383141 2.574258144
F(x) = 0.3441152698
Improvement is 0.01464599787
∇F(x) = 0.7196625374 0.1406434803

Q (approx. of Hessian inverse)

0.04700909863	0.09177565874
0.09177565874	0.6066786058

Actual Hessian inverse

0.5818306937	1.841358161
1.841358161	6.32746822

Search direction is ¯0.04673833525 ¯0.151372894
Optimal stepsize is 9.360119345
Step is Δx= ¯0.437476396 ¯1.416868353,
with magnitude 1.482869356
Δgradient = 0.2727649394 ¯0.4474868063

Q updated: Q ← Q + A + B, where matrices A,B =

0.3718379161	1.204282975
1.204282975	3.900348571

¯0.007777870003	¯0.06786231974
¯0.06786231974	¯0.5921022644

Iteration 4

x = 1.144906745 1.157389791
F(x) = 0.04453617137
∇F(x) = 0.9924274768 ¯0.3068433261

Q (approx. of Hessian inverse)

$$\begin{bmatrix} 0.411069144 & 1.228196314 \\ 1.228196314 & 3.914924913 \end{bmatrix}$$

Actual Hessian inverse

$$\begin{bmatrix} 0.3826013341 & 0.876085695 \\ 0.8760856959 & 2.506072844 \end{bmatrix}$$

Search direction is ¯0.03109247191 ¯0.01762718775
Optimal stepsize is 3.5355748
Step is Δx= ¯0.1099297602 ¯0.0623222408,
with magnitude 0.1263669809

Δgradient¯= ¯1.021376702 0.3546237095

Q updated: Q ← Q + A + B, where matrices A,B =

$$\begin{bmatrix} 0.1340066473 & 0.07597209829 \\ 0.0759720982 & 0.04307069712 \end{bmatrix} \quad \begin{bmatrix} -0.00782901597 & -0.0667960420 \\ -0.06679604207 & -0.5698942564 \end{bmatrix}$$

Iteration 5

x = 1.034976985 1.09506755
F(x) = 0.001794130712
Improvement is 0.04274204066
∇F(x) = ¯0.02894922502 0.0477803834

Q (approx. of Hessian inverse)

$$\begin{bmatrix} 0.5372467761 & 1.237372371 \\ 1.237372371 & 3.388101353 \end{bmatrix}$$

Actual Hessian inverse

$$\begin{bmatrix} 0.5250889514 & 1.086909959 \\ 1.086909959 & 2.749853584 \end{bmatrix}$$

Search direction is ¯0.04356924846 ¯0.1260638105
Optimal stepsize is 0.7330121416
Step is Δx= ¯0.03193678812 ¯0.09240630368,
with magnitude 0.09776954226

Δgradient = 0.04878486626 ¯0.05463716156

Q updated: Q ← Q + A + B, where matrices A,B =

$$\begin{bmatrix} 0.2921858788 & 0.8454142898 \\ 0.8454142898 & 2.44613232 \end{bmatrix} \quad \begin{bmatrix} -0.357283507 & -1.076685326 \\ -1.076685326 & -3.244625819 \end{bmatrix}$$

Iteration 6

x = 1.003040197 1.002661247

F(x) = 0.00002099664646
Improvement is 0.001773134066

∇F(x) = 0.01983564124 ¯0.006856778164

Q (approx. of Hessian inverse)   Actual Hessian inverse

$$\begin{bmatrix} 0.4721491479 & 1.006101334 \\ 1.006101334 & 2.589607854 \end{bmatrix} \begin{bmatrix} 0.4965949585 & 0.9962094096 \\ 0.9962094096 & 2.498476164 \end{bmatrix}$$

Search direction is ¯0.002466767447 ¯0.002200298527
Optimal stepsize is 1.242655471
Step is Δx= ¯0.003065342064 ¯0.002734213002,
with magnitude 0.004107583561

Δgradient = ¯0.01979523147 0.006811426577

Q updated: Q ← Q + A + B, where matrices A,B =

$$\begin{bmatrix} 0.2234279599 & 0.1992924836 \\ 0.1992924836 & 0.1777642066 \end{bmatrix} \quad \begin{bmatrix} ^{-}0.1836761831 & ^{-}0.167746983 \\ ^{-}0.167746983 & ^{-}0.1531992326 \end{bmatrix}$$

Iteration 7

x = 0.9999748544 0.9999270337

F(x) = 1.146490964E¯9
Improvement is 0.00002099549997

∇F(x) = 0.00004040976709 ¯0.00004535158715

Q (approx. of Hessian inverse)   Actual Hessian inverse

$$\begin{bmatrix} 0.5119009247 & 1.037646835 \\ 1.037646835 & 2.614172828 \end{bmatrix} \quad \begin{bmatrix} 0.4999773252 & 0.999929506 \\ 0.999929506 & 2.499808725 \end{bmatrix}$$

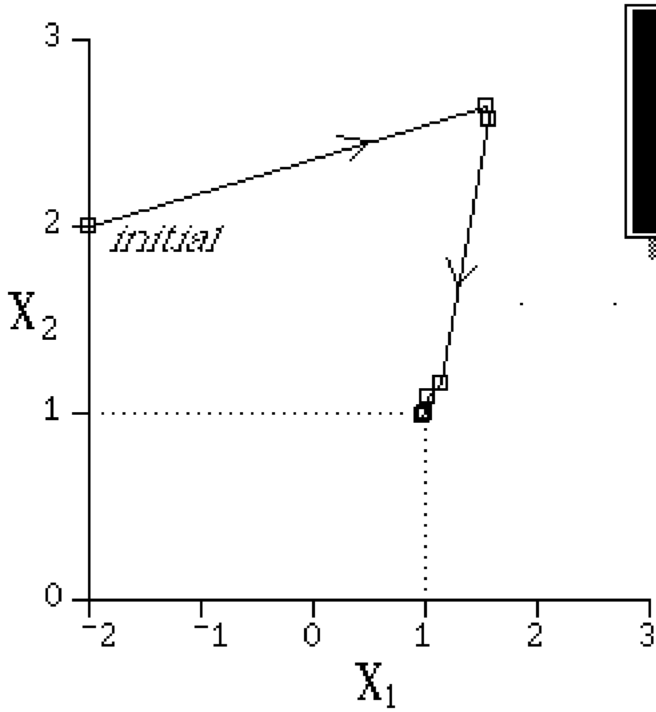
Convergence criterion satisfied:

Gradient

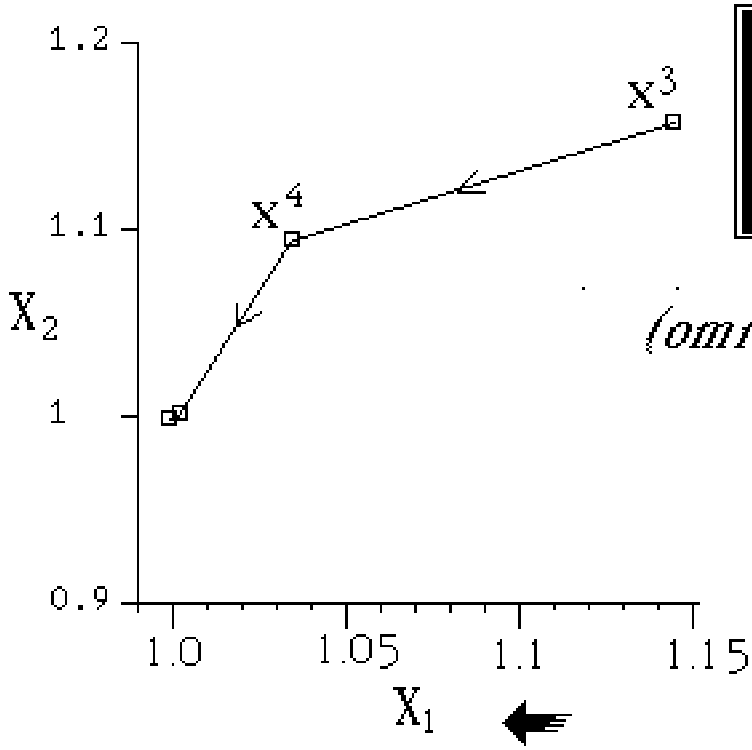
*** CONVERGED ***

Solution found is 0.9999748544 0.9999270337
where F is 1.146490964E¯9

```
*** CONVERGED ***

Solution found is 0.9999748544 0.9999270337
where F is 1.146490964E¯9
and ∇F is 0.00004040976709 ¯0.00004535158715

# iterations = 7
# function evaluations= 60
# gradient evaluations= 34
Elapsed CPU time: 77.6 seconds
```

Davidon-Fletcher-Powell (DFP) Algorithm



Davidon-Fletcher-Powell (DFP) Algorithm

(omitting x^0, x^1, & x^2)

Powell's Method

In the example computation which follows, x= (3,3) was used as a starting point, rather than (-2, 2).
(My APL code terminated prematurely when (-2,2) was used to start the algorithm.)

Powell's Method

Tolerances:
  1-dimensional search: 0.001
  stopping criterion: |ΔF| ≤ 0.00001

Cycle 1

```
Cycle Starts at x = 3 3
Where F(x) = 40, with Directions:
                        1) 1 0
                        2) 0 1
x = 1.67268 3
   F(x) = 0.49336, improvement = 39.5066   ← search in direction 1

x = 1.67268 2.79784
   F(x) = 0.452492, improvement = 0.0408677   ← search in second direction

New direction: ¯1.32732 ¯0.202158

x = 1.60911 2.78816   F(x) = 0.410587   ← search in new direction
```


Cycle 2

```
Cycle Starts at x = 1.60911 2.78816
Where F(x) = 0.410587, with Directions:
                    1)  0          1
                    2) ¯1.32732  ¯0.202158

Change in F during previous cycle was ΔF = 39.5894

x = 1.60911 2.58923
   F(x) = 0.371014, improvement = 0.0395727   ← search in direction 1

x = 1.54601 2.57962
   F(x) = 0.334028, improvement = 0.0369862   ← search in second direction

New direction: ¯0.063099 ¯0.208539

x = 1.1067 1.12771   F(x) = 0.0208058   ← search in new direction
```

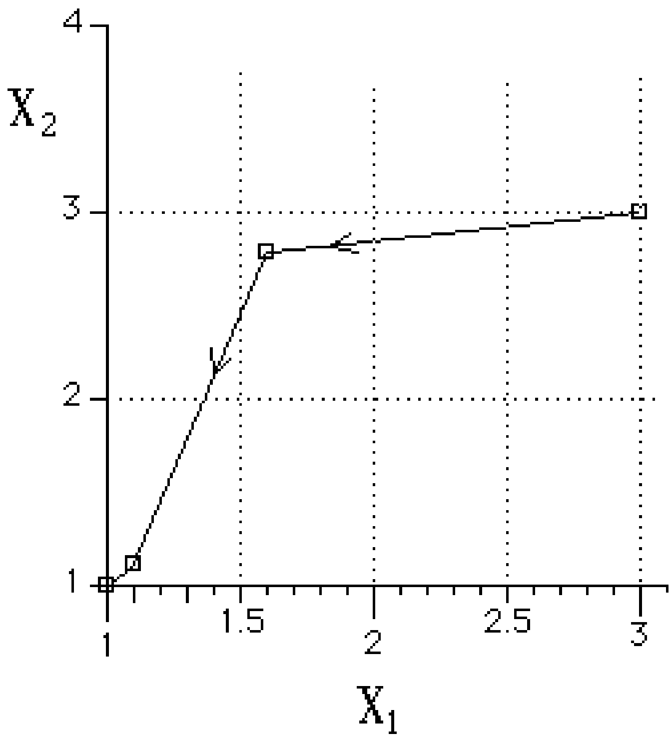
Cycle 3

```
Cycle Starts at x = 1.1067 1.12771
Where F(x) = 0.0208058, with Directions:
                    1) ¯1.32732  ¯0.202158
                    2) ¯0.063099 ¯0.208539
Change in F during previous cycle was ΔF = 0.389781

x = 1.04624 1.11851
   F(x) = 0.00270882, improvement = 0.0180969
x = 1.01701 1.02192
   F(x) = 0.00044289, improvement = 0.00226593

New direction: ¯0.0896872 ¯0.105796

x = 1.00088 1.00289   F(x) = 2.05195E¯6
```


Cycle 4

```
Cycle Starts at x = 1.00088 1.00289
Where F(x) = 2.05195E¯6, with Directions:

                    1) ¯0.063099  ¯0.208539
                    2) ¯0.0896872 ¯0.105796

Change in F during previous cycle was ΔF = 0.0208037

x = 1.00001 1.00001
   F(x) = 9.0524E¯11, improvement = 2.05186E¯6
x = 1.00018 1.00021
   F(x) = 5.29529E¯8, improvement = ¯5.28624E¯8

New direction: ¯0.000701781 ¯0.00268185

x = 1.0002 1.00029   F(x) = 5.1093E¯8
```

```
*** CONVERGED ***

Solution found is 1.0002 1.00029
where F is 5.1093E¯8

# iterations = 4
# function evaluations= 123
Elapsed CPU time: 49.1 seconds
```

Powell's (version 1) method

Algorithm: Powell's (version 1) method

i	F	ΔF	logΔF
1	4E1	4E1	1.60206E0
2	4.10587E⁻1	4.10587E⁻1	⁻3.86595E⁻1
3	2.08058E⁻2	2.08058E⁻2	⁻1.68182E0
4	2.05195E⁻6	2.05195E⁻6	⁻5.68783E0
5	5.1093E⁻8	5.1093E⁻8	⁻7.29164E0