

# Queueing Networks-- An Introduction

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M/M/2/2 Queueing System



Two Tandem Servers w/o Queues



Two Tandem Servers w/intervening Queue



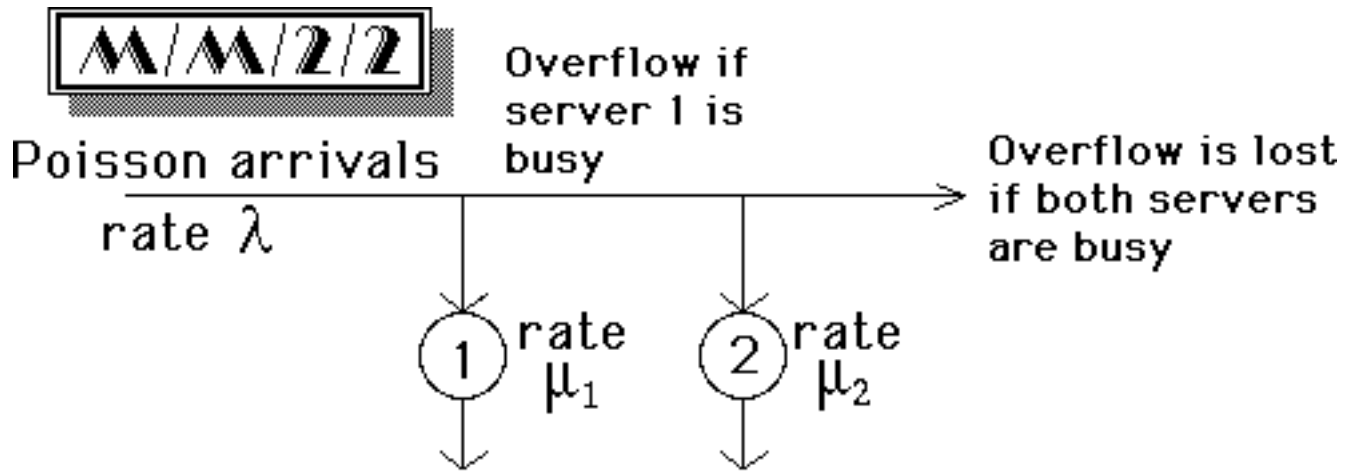
M/M/1 System with feedback



Two Cyclic Queues



Two Tandem Queues w/infinite capacity



*We want to compute:*

- *steady-state distribution*
- *fraction of customers lost*
- *utilization of each server*

↻

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**States**  $(n_1, n_2)$  where  $n_i = \begin{cases} 1 & \text{if server } i \\ & \text{is busy} \\ 0 & \text{otherwise} \end{cases}$

(0,0)

(1,0)

(0,1)

(1,1)

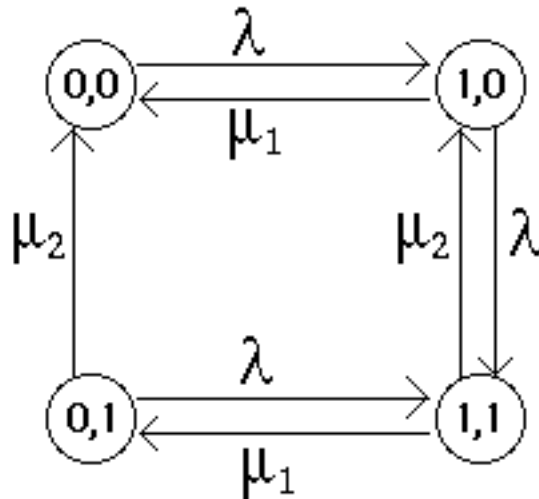
*What are the transition rates?*

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**States**

$(n_1, n_2)$  where

$$n_i = \begin{cases} 1 & \text{if server } i \\ & \text{is busy} \\ 0 & \text{otherwise} \end{cases}$$



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**Transition rate matrix**

to \ from	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	-0.2	0.2	0	0
(1,0)	0.33333	-0.53333	0	0.2
(0,1)	0.25	0	-0.45	0.2
(1,1)	0	0.25	0.3333	-0.5833

parameters:

$$\lambda = 1/5$$

$$\mu_1 = 1/3$$

$$\mu_2 = 1/4$$

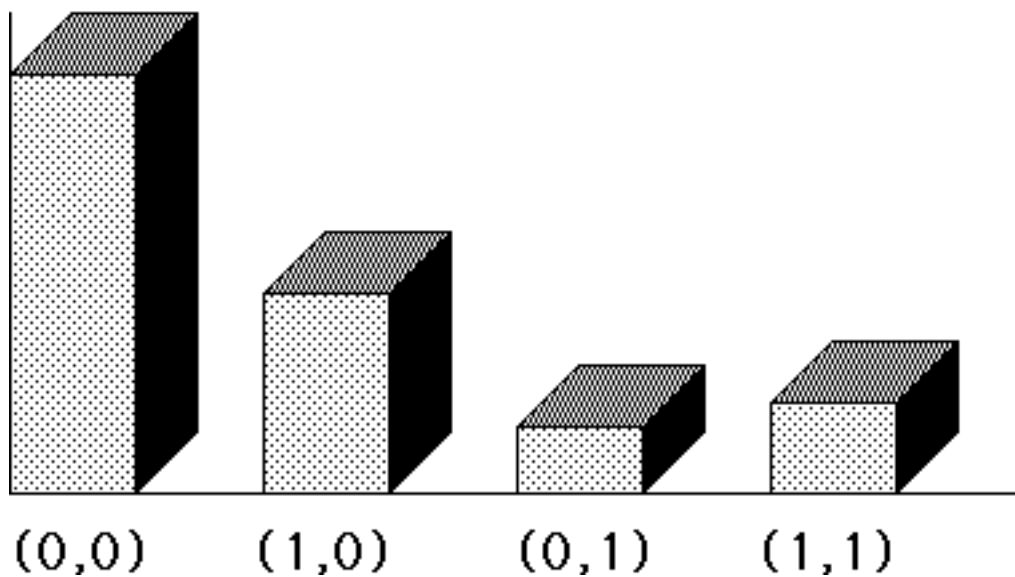
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### Steadystate Distribution

i	state	Pi
1	(0,0)	0.537536
2	(1,0)	0.256924
3	(0,1)	0.0874636
4	(1,1)	0.118076

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### Steady-State Distribution



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0.191205 <sup>PI</sup> 0.449331 0.152964 0.206501

Average # in System

$$L = \sum_{i \in S} n_i \pi_i$$

0.580539 <sup>PI+.x0 1 1 2</sup>

Average Arrival Rate

$$\bar{\lambda} = \sum_{i \in S} \lambda_i \pi_i$$

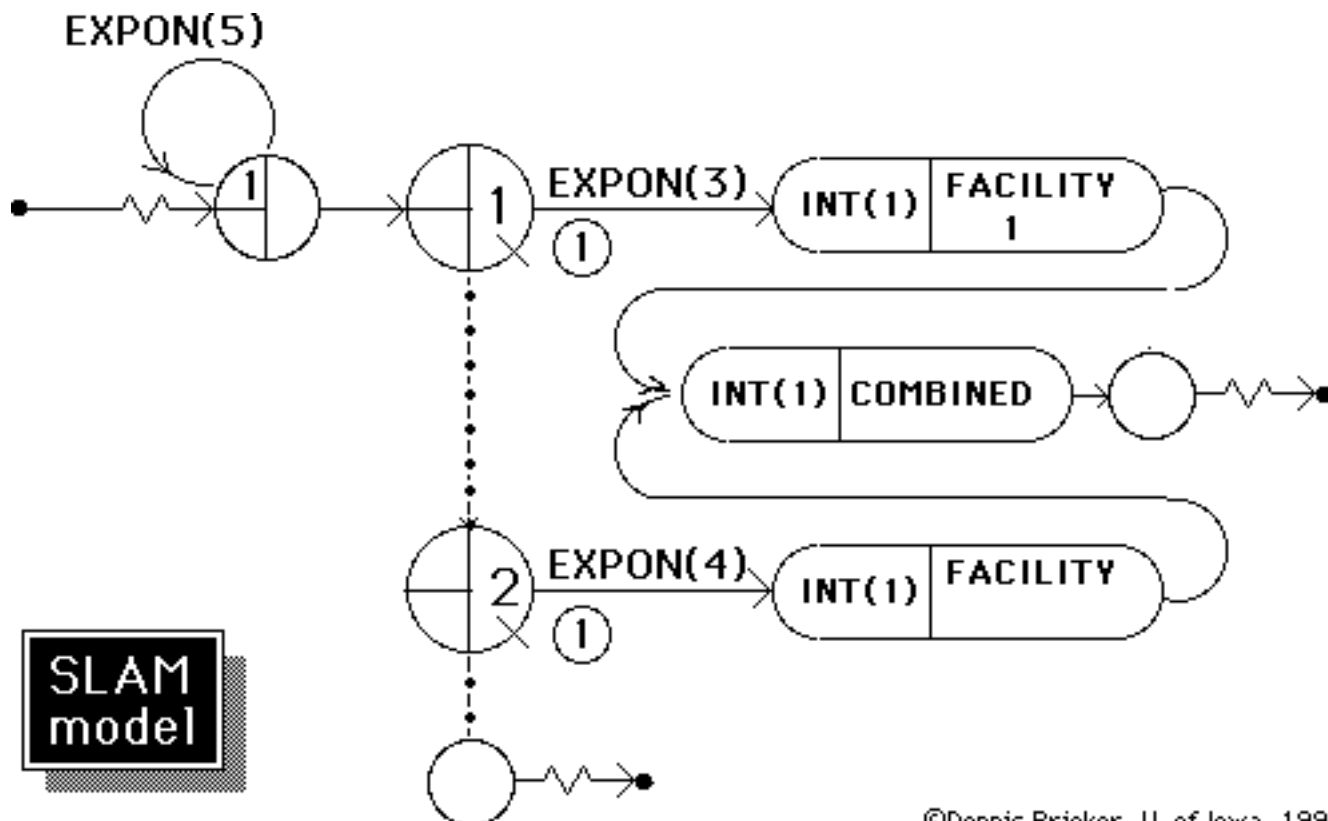
0.176385 <sup>PI+.x .2 .2 .2 0</sup>

Average Time in System

$$W = L / \bar{\lambda}$$

3.29132 <sup>(PI+.x0 1 1 2) ÷ PI+.x .2 .2 .2 0</sup>

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```

GEN, BRICKER, M_M_2_2, 4/14/93, 1, Y, Y, Y/N, Y, Y, 72;
LIM, 2, 2, 100;
INIT, , 3700;
MONTR, CLEAR, 100
NETWORK;
    CREATE, EXPON(5.0), , 1;
Q1  QUEUE(1), , 0, BALK(Q2);
    ACTIVITY/1, EXPON(3.0); FACILITY 1
    COLCT(1), INT(1), TIME_IN_SYS_1;
    ACT, , , C3;
Q2  QUEUE(2), , 0, BALK(T);
    ACTIVITY/2, EXPON(4.0); FACILITY 2
    COLCT(2), INT(1), TIME_IN_SYS_2;
C3  COLCT(3), INT(1), TIME_IN_SYS_3;
T   TERM;
    END;
FIN:
    
```

**SLAM code**

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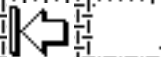
**\*\*STATISTICS FOR VARIABLES BASED ON OBSERVATION\*\***

	MEAN VALUE	STANDARD DEVIATION	COEFF. OF VARIATION	MINIMUM VALUE	MAXIMUM VALUE	NO. OF OBS
TIME_IN_SYS_1	0.288E+01	0.291E+01	0.101E+01	0.110E-02	0.352E+02	459
TIME_IN_SYS_2	0.367E+01	0.386E+01	0.105E+01	0.234E-01	0.253E+02	186
TIME_IN_SYS_3	0.311E+01	0.323E+01	0.104E+01	0.110E-02	0.352E+02	645

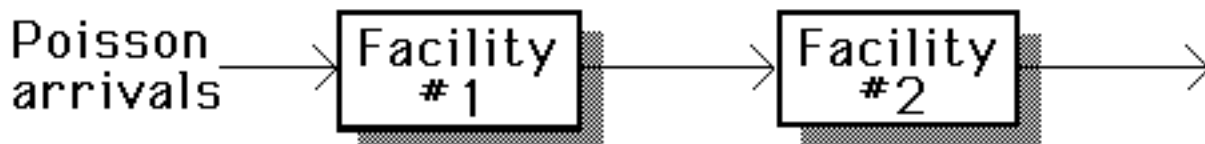
**\*\*SERVICE ACTIVITY STATISTICS\*\***

ACT NUM	ACT START	LABEL OR NODE	SER CAP	AVERAGE UTIL	STD DEV	CUR UTIL	AVERAGE BLOCK	MAX TME/SER	IDL TME/SER	MAX BSY	ENT CNT
1	FACILITY	1	1	0.368	0.48	0	0.00	27.63	35.22	459	
2	FACILITY	2	1	0.187	0.39	0	0.00	94.70	25.27	186	

Comparison	Simulation	Markov Chain Analysis
Average # in System	0.555	0.581
Average Time in System	3.11	3.29



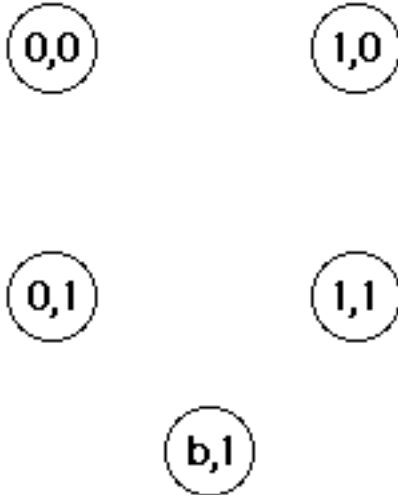
## 2 Tandem Servers, with no queues



- 2 identical servers, with exponentially dist'd service times
- No queues allowed in front of either server
- Server 1 is "blocked" whenever it has completed service while server 2 is busy
- Arrivals at #1 are turned away when it is busy or blocked ↻

**States**

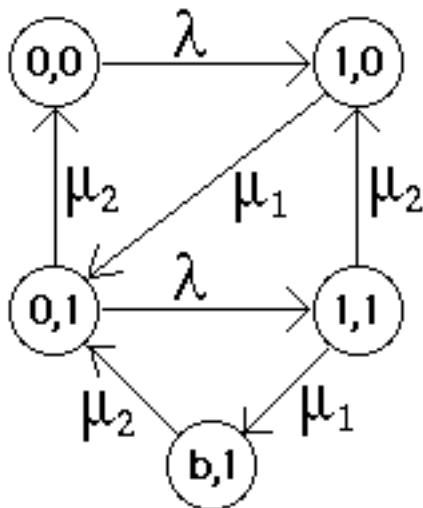
(x,y) where  $x = \begin{cases} 0 & \text{if \#1 is idle} \\ 1 & \text{if \#1 is busy} \\ b & \text{if \#1 is blocked} \end{cases}$   
 $y = \begin{cases} 0 & \text{if \#2 is idle} \\ 1 & \text{if \#2 is busy} \end{cases}$



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**States**

(x,y) where  $x = \begin{cases} 0 & \text{if \#1 is idle} \\ 1 & \text{if \#1 is busy} \\ b & \text{if \#1 is blocked} \end{cases}$   
 $y = \begin{cases} 0 & \text{if \#2 is idle} \\ 1 & \text{if \#2 is busy} \end{cases}$



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Tandem Servers w/o queues

Transition rate matrix

parameters:

$$\lambda = 4$$

$$\mu_1 = 4$$

$$\mu_2 = 5$$

		to				
		1	2	3	4	5
f r o m	1	-4	4	0	0	0
	2	0	-4	4	0	0
	3	5	0	-9	4	0
	4	0	5	0	-9	4
	5	0	0	5	0	-5

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Tandem Servers w/o queues

Steadystate Distribution

i	state	Pi
1	(0,0)	0.257437
2	(1,0)	0.371854
3	(0,1)	0.20595
4	(1,1)	0.0915332
5	(b,1)	0.0732265

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Average # in System

$$L = \sum_{i \in S} n_i \pi_i \quad \text{PI+.x 0 1 1 2 2} \\ 0.907323$$

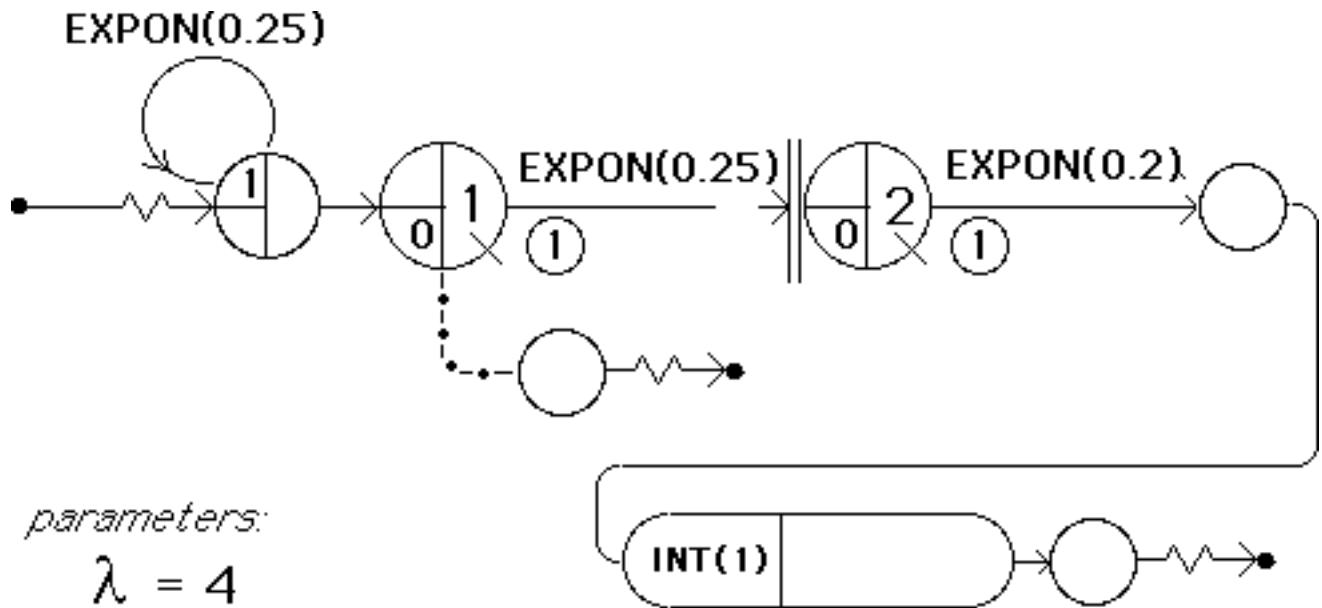
Average Arrival Rate

$$\bar{\lambda} = \sum_{i \in S} \lambda_i \pi_i \quad \text{PI+.x 4 0 4 0 0} \\ 1.85355$$

Average Time in System

$$W = L / \bar{\lambda} \quad \text{(PI+.x 0 1 1 2 2) \div PI+.x 4 0 4 0 0} \\ 0.489506$$

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parameters:

$$\lambda = 4 \\ \mu_1 = 4 \\ \mu_2 = 5$$

**SLAM  
model**

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```

GEN,BRICKER,TANDEM,4/14/93,1,Y,Y,Y/N,Y,Y,72;
LIM,2,2,100;
INIT,,3700;
MONTR,CLEAR,100
NETWORK;
    CREATE,EXPON(0.25),,1;
Q1    QUEUE(1),,0,BALK(T);
      ACTIVITY/1,EXPON(0.25); FACILITY 1
Q2    QUEUE(2),,0,BLOCK;
      ACTIVITY/2,EXPON(0.20); FACILITY 2
      COLCT(1),INT(1),TIME_IN_SYS;
T     TERM;
      END;
FIN;
    
```

**SLAM code**

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**\*\*STATISTICS FOR VARIABLES BASED ON OBSERVATION\*\***

	MEAN VALUE	STANDARD DEVIATION	COEFF. OF VARIATION	MINIMUM VALUE	MAXIMUM VALUE	NO. OF OBS
TIME_IN_SYS	0.481E+00	0.322E+00	0.671E+00	0.586E-02	0.287E+01	6708

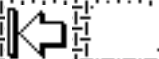
**\*\*FILE STATISTICS\*\***

FILE NUMBER	LABEL/TYPE	AVERAGE LENGTH	STANDARD DEVIATION	MAXIMUM LENGTH	CURRENT LENGTH	AVERAGE WAIT TIME
1	Q1 QUEUE	0.000	0.000	0	0	0.000
2	Q2 QUEUE	0.000	0.000	0	0	0.000

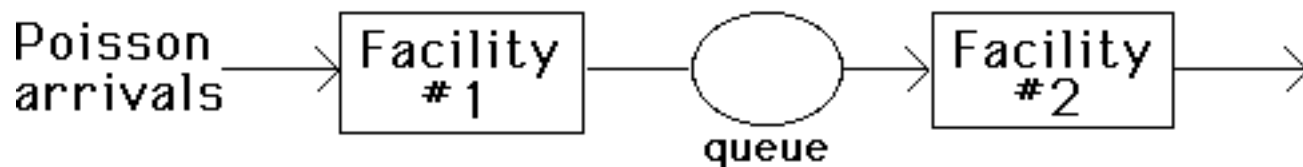
**\*\*SERVICE ACTIVITY STATISTICS\*\***

ACT NUM	ACT START	LABEL OR NODE	SER CAP	AVERAGE UTIL	STD DEV	CUR UTIL	AVERAGE BLOCK	MAX TME/SER	IDL TME/SER	MAX BSY TME/SER	ENT CNT
1	FACILITY 1	1	1	0.463	0.50	1	0.07	2.43	2.42	6708	
2	FACILITY 2	1	1	0.364	0.48	0	0.00	2.78	1.71	6708	

Comparison	Simulation	Markov Chain Analysis
Average # in System	0.827	0.907
Average Time in System	0.481	0.489



*Suppose that we add space for 1 customer to wait between the two facilities...*

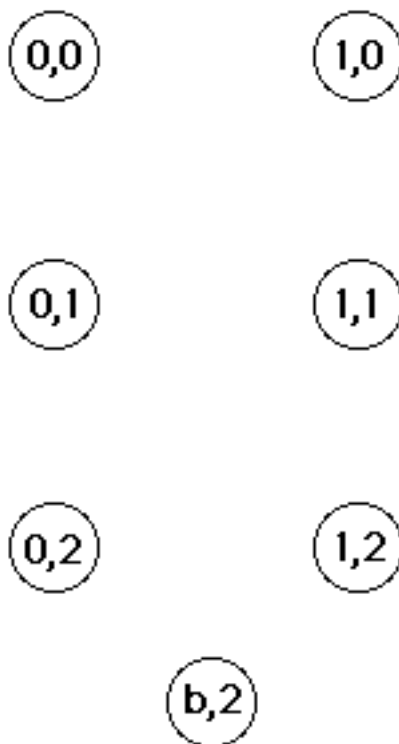


# States

(x,y) where

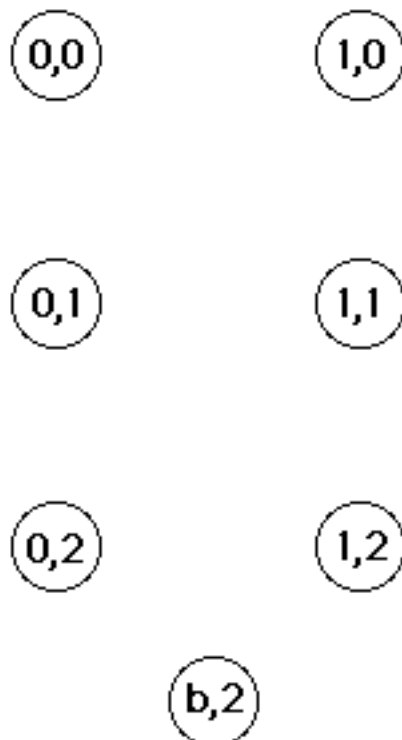
$$x = \begin{cases} 0 & \text{if \#1 is idle} \\ 1 & \text{if \#1 is busy} \\ b & \text{if \#1 is blocked} \end{cases}$$

$$y = \begin{cases} 0 & \text{if \#2 is idle} \\ 1 & \text{if \#2 is busy} \\ 2 & \text{if customer waits} \\ & \text{between} \end{cases}$$

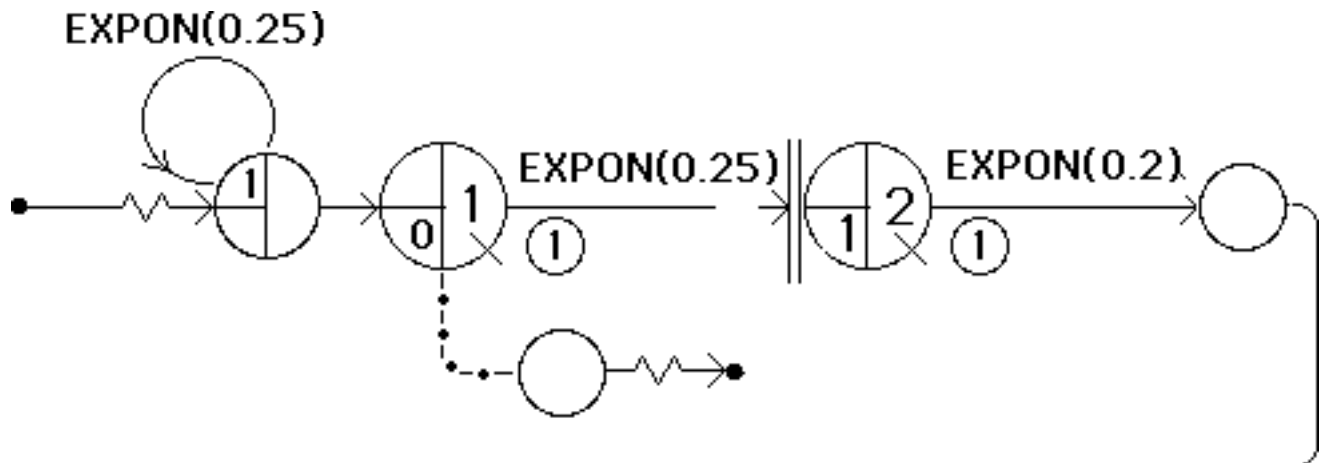


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# Transition Diagram



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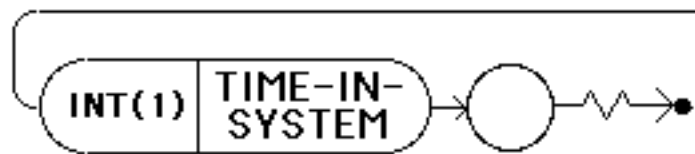


parameters:

$$\lambda = 4$$

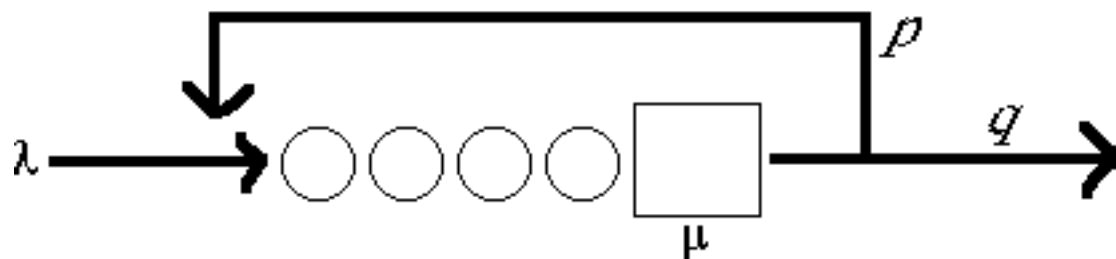
$$\mu_1 = 4$$

$$\mu_2 = 5$$



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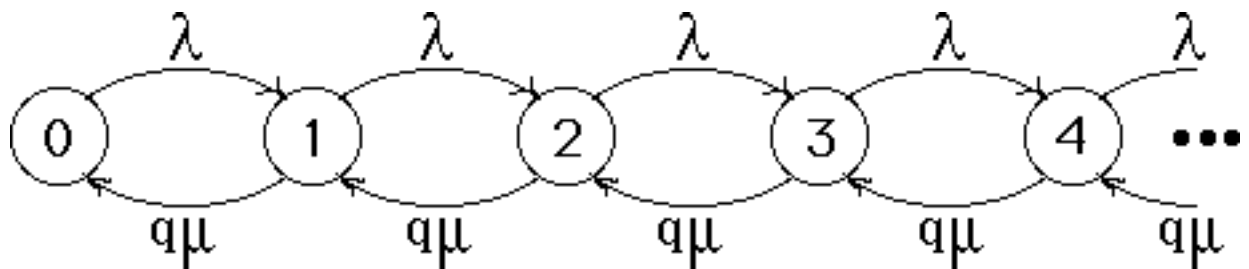
**M/M/1 System with feedback**



A customer, when service is complete, will depart with probability  $q$  and return to end of queue with probability  $p=1-q$

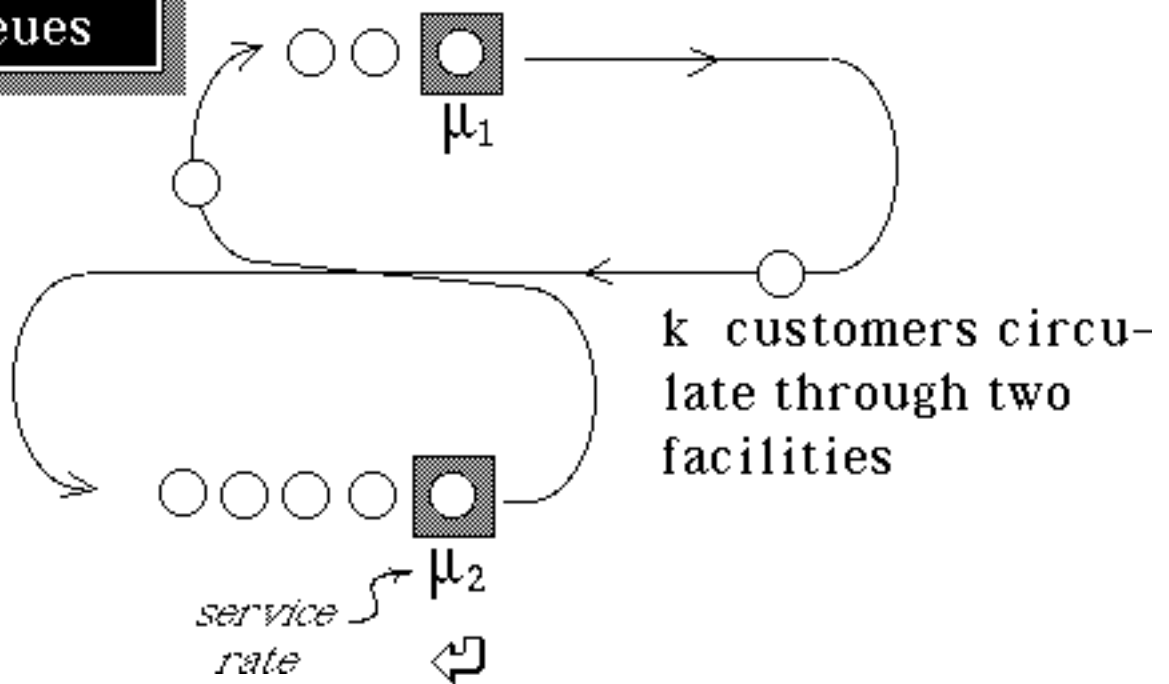


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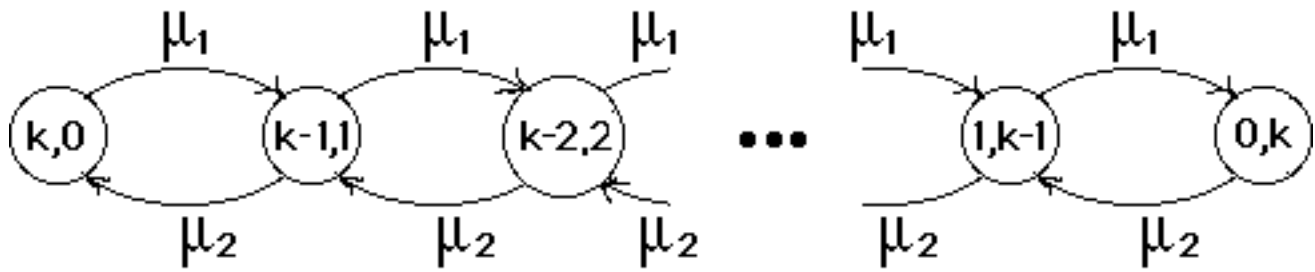
**Two Cyclic Queues**



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**States**

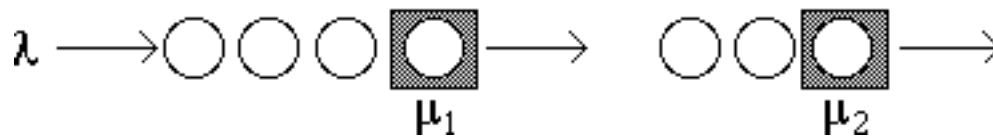
$(i,j)$  where  
 $i = \#$  customers in subsystem 1  
 $j = \#$  customers in subsystem 2



*Is this a birth-death process?  
 Compare with  $M/M/1/k$  queueing system!*

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**Two Tandem Queues with infinite capacity**

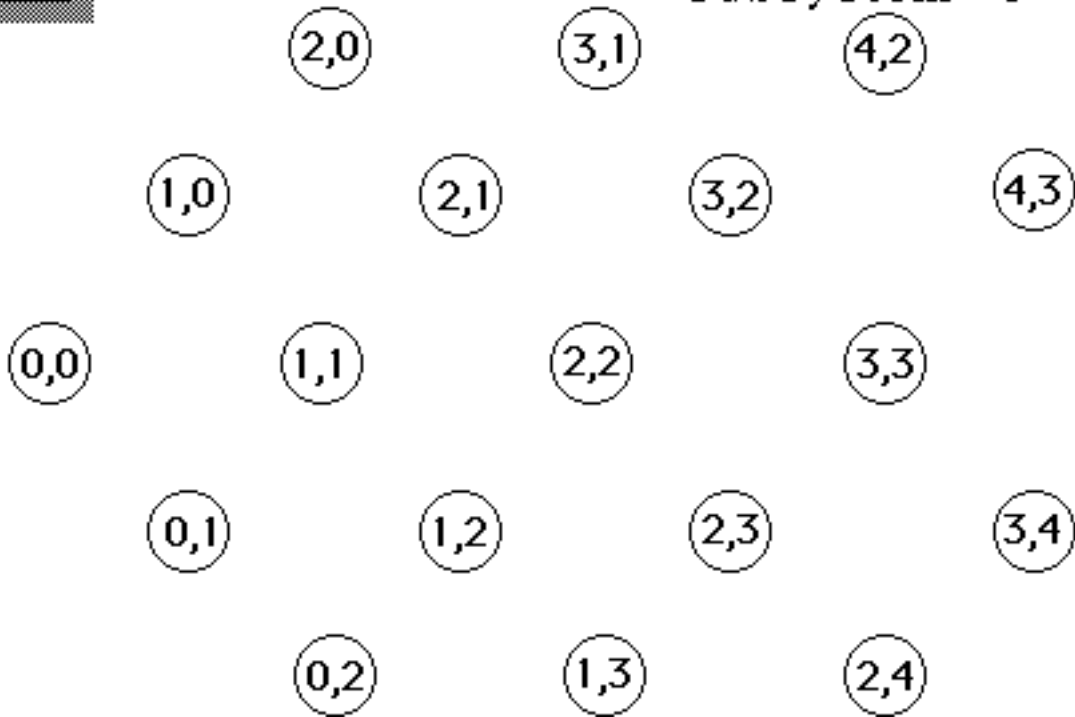


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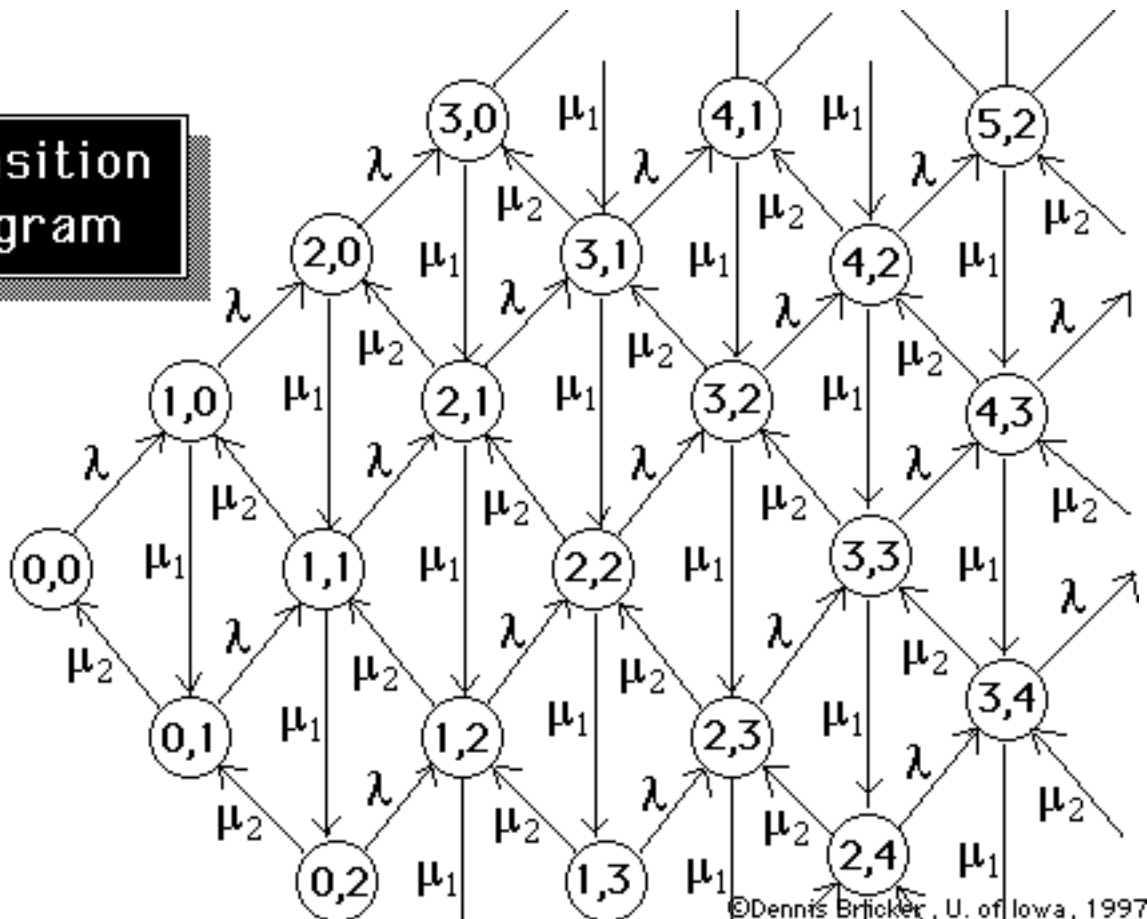
**States**

$(m_1, m_2)$  where  $m_i = \#$ customers in subsystem #i



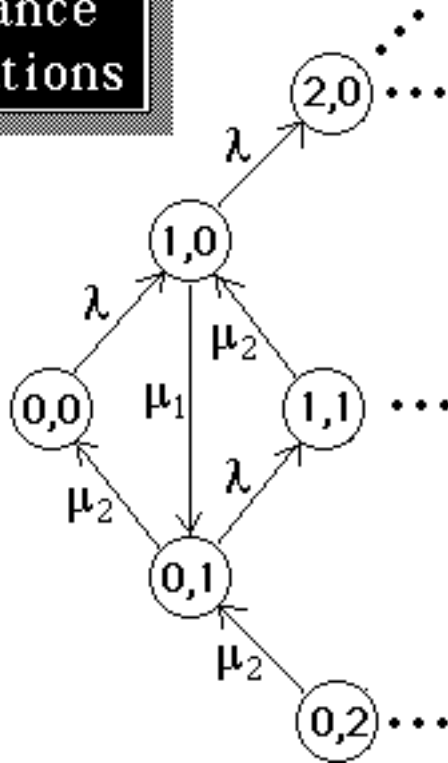
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**Transition Diagram**



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**Balance Equations**



for state 0,0

$$\lambda \pi_{00} = \mu_2 \pi_{01}$$

for state 1,0

$$(\lambda + \mu_1) \pi_{10} = \lambda \pi_{00} + \mu_2 \pi_{11}$$

for state 0,1

$$(\lambda + \mu_2) \pi_{01} = \mu_1 \pi_{10} + \mu_2 \pi_{02}$$

etc.

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**Balance Equations**

$$\lambda \pi_{00} = \mu_2 \pi_{01}$$

$$(\lambda + \mu_1) \pi_{10} = \lambda \pi_{00} + \mu_2 \pi_{11}$$

$$(\lambda + \mu_2) \pi_{01} = \mu_1 \pi_{10} + \mu_2 \pi_{02}$$

$$(\lambda + \mu_1) \pi_{20} = \lambda \pi_{10} + \mu_2 \pi_{21}$$

$$(\lambda + \mu_1 + \mu_2) \pi_{11} = \lambda \pi_{01} + \mu_1 \pi_{20} + \mu_2 \pi_{12}$$

$$(\lambda + \mu_2) \pi_{02} = \mu_1 \pi_{11} + \mu_2 \pi_{03}$$

⋮

*We get infinitely many equations in infinitely many unknowns!*

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*Claim: these balance equations are satisfied by*

$$\pi_{m_1, m_2} = \pi_{m_1}^1 \times \pi_{m_2}^2$$

*where*

$$\pi_{m_i}^i = (1 - \rho_i) \rho_i^{m_i}, \quad \rho_i = \lambda / \mu_i$$

*is the steady-state distribution of the M/M/1 queue*

That is,

$$\begin{aligned} & P\{m_1 \text{ at station 1 \& } m_2 \text{ at station 2}\} \\ &= P\{m_1 \text{ at station 1}\} \times P\{m_2 \text{ at station 2}\} \end{aligned}$$

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*Substituting into*

$$\lambda \pi_{00} = \mu_2 \pi_{01}$$

$$\begin{aligned} \pi_{m_1, m_2} &= \pi_{m_1}^1 \times \pi_{m_2}^2 \\ \pi_{m_i}^i &= (1 - \rho_i) \rho_i^{m_i}, \quad \rho_i = \lambda / \mu_i \end{aligned}$$

*yields* 
$$\lambda \left(1 - \frac{\lambda}{\mu_1}\right) \left(1 - \frac{\lambda}{\mu_2}\right) = \mu_2 \left(1 - \frac{\lambda}{\mu_1}\right) \left(1 - \frac{\lambda}{\mu_2}\right) \frac{\lambda}{\mu_2}$$

*Substituting into* 
$$(\lambda + \mu_1) \pi_{10} = \lambda \pi_{00} + \mu_2 \pi_{11}$$

*yields*

$$\begin{aligned} (\lambda + \mu_1) \left(1 - \frac{\lambda}{\mu_1}\right) \frac{\lambda}{\mu_1} \left(1 - \frac{\lambda}{\mu_2}\right) &= \\ \lambda \left(1 - \frac{\lambda}{\mu_1}\right) \left(1 - \frac{\lambda}{\mu_2}\right) + \mu_2 \left(1 - \frac{\lambda}{\mu_1}\right) \frac{\lambda}{\mu_1} \left(1 - \frac{\lambda}{\mu_2}\right) \frac{\lambda}{\mu_2} & \end{aligned}$$

*etc.*

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