# Portfolio Optimization via Quadratic Programming

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A fixed sum of money $F$ is to be allocated among $n$ investments, each of which has a known history of returns during the previous $p$ periods

$r_{ik}$ = return per dollar invested in investment #i during period k, i=1,...n; k=1,...p

$x_i$ = amount of money to be allocated to investment #i

**EXAMPLE**

$n=3$, $p=5$

| Investment | Year 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| #1 | 10% | 4% | 12% | 13% | 6% |
| #2 | 6% | 9% | 6% | 5% | 9% |
| #3 | 17% | 1% | 11% | 19% | 2% |

$\underbrace{\phantom{xx}}_{r_{ik}}$

Annual Return

F = $10,000 available for investment



Let

$x_i$ = amount of money to be allocated to investment #i

Assuming that past history is indicative of future performance, the expected annual return will be

$$E = \sum_{i=1}^{n} e_i x_i$$

where

$$e_i = \frac{1}{p}\sum_{k=1}^{p} r_{ik}$$

## Expected Annual Returns

$$\begin{cases} e_1 = \frac{1}{5}(0.10 + 0.04 + 0.12 + 0.13 + 0.06) = 0.09 \\ e_2 = \frac{1}{5}(0.06 + 0.09 + 0.06 + 0.05 + 0.09) = 0.07 \\ e_3 = \frac{1}{5}(0.17 + 0.10 + 0.11 + 0.19 + 0.02) = 0.10 \end{cases}$$

The expected total annual return from the investments will be

$$\boxed{E = 0.09\,x_1 + 0.07\,x_2 + 0.10\,x_3}$$



$$\boxed{E = 0.09\,x_1 + 0.07\,x_2 + 0.10\,x_3}$$

If we wish to maximize the expected return, then we would invest the total available funds in investment #3, which has the highest expected return.

Looking at the past history, however, we see a greater variability in the return provided by investment #3:

| | | | | | |
|---|---|---|---|---|---|
| #1 | 10% | 4% | 12% | 13% | 6% |
| #2 | 6% | 9% | 6% | 5% | 9% |
| #3 | 17% | 1% | 11% | 19% | 2% |

The Variance of the total annual return, based upon past performance, is

$$\boxed{V = \frac{1}{p}\sum_{k=1}^{p}\Big[\underbrace{\sum_{i=1}^{n} r_{ik}\, x_i}_{\substack{\text{total return}\\ \text{in year } k\\ \text{if you had}\\ \text{invested } x}} - E\Big]^2}$$

*expected return per year*



Substitute $\boxed{E = \sum_{i=1}^{n} e_i\, x_i}$ into $\boxed{V = \frac{1}{p}\sum_k\left(\sum_i r_{ik}\, x_i - E\right)^2}$

to get $V = \frac{1}{p}\sum_k\left(\sum_i r_{ik}\, x_i - \sum_i e_i\, x_i\right)^2$

$$= \frac{1}{p}\sum_k\left(\sum_i [r_{ik} - e_i]\, x_i\right)^2$$

$$= \frac{1}{p}\sum_k\sum_i\sum_j (r_{ik} - e_i)\,(r_{jk} - e_j)\, x_i x_j$$

$V = \sum_i\sum_j \sigma_{ij}^2\, x_i x_j$ where $\sigma_{ij}^2 = \frac{1}{p}\sum_k (r_{ik} - e_i)\,(r_{jk} - e_j)$

$$= \frac{1}{p}\sum_k r_{ik} r_{jk} - \frac{1}{p^2}\left(\sum_k r_{ik}\right)\left(\sum_k r_{jk}\right)$$

Thus, the variance of the total return is the quadratic function $x^T C x$

where C is the covariance matrix with entries $\sigma_{ij}^2$



In our example, the covariance matrix is

$$C = \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} \times 10^{-4}$$

If we invest all \$10,000 in #3 in order to maximize the expected return, i.e., $x = (0,0,10^4)$ the variance of the annual return will be

$$x^T C x = 55.2 \times 10^{-4} \times 10^4 \times 10^4 = 55.2 \times 10^4$$

i.e., the standard deviation will be \$743, or 74% of the expected return (\$1000)! ... a "risky" investment.

Suppose that we are "risk-averse" and are satisfied with an 8% annual rate of return, but wish to minimize the variance of the total return.

Then we must solve:

**Quadratic Programming Problem**

$$\text{Minimize } x^T \begin{bmatrix} 12 & -5.6 & 23 \\ -5.6 & 2.8 & -12 \\ 23 & -23 & 55.2 \end{bmatrix} x$$

subject to

$$x_1 + x_2 + x_3 \le 10000$$
$$0.09x_1 + 0.07x_2 + 0.10\, x_3 \ge 800$$
$$x_j \ge 0,\ j=1,2,3$$



Optimal solution: $x = (5000,5000,0)$

*i.e., invest half of the total in each of investments #1 & #2, and nothing in #3 which yields the greatest expected return!*

The expected return will be 8% ($800), with a variance of $0.009\text{x}10^6$, i.e., a standard deviation of $94.87.