EXAMPLE
Complementary
Pivoting

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Minimize $x_1^2 - 2x_1x_2 + x_2^2 - 4x_1 - 6x_2$

subject to

$$2x_1 + x_2 \le 2$$
$$-x_1 + x_2 \le 4$$
$$x_1 \ge 0, x_2 \ge 0$$

EXAMPLE

Minimize $\frac{1}{2}x^\top \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} x + \begin{bmatrix} -4, & -6 \end{bmatrix} x$

subject to $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x \le \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$x \ge 0$$

Tableau (before adding artificial variable)

1	2	3	4	5	6	7	8	b
2	1	0	0	1	0	0	0	2
-1	1	0	0	0	1	0	0	4
2	-2	2	-1	0	0	-1	0	4
-2	2	1	1	0	0	0	-1	6

These represent the K.T. conditions:
$AX \le B$, i.e. $AX + Y = B$
$HX + (\wr A)U + C \ge 0$, i.e. $HX + (\wr A)U - V = -C$

(In addition, we must impose:
Complementary Slackness: $XV = 0$, $YU = 0$
Nonnegativity: $X \ge 0$, $Y \ge 0$, $U \ge 0$, $V \ge 0$)

Variable numbers:
    X: 1 2
    Y: 5 6 (slack variables for primal constraints)

    U: 3 4 (multipliers for $Ax \le b$ constraints)
    V: 7 8 (multipliers for $x \ge 0$ constraints)

1	2	3	4	5	6	7	8	b
2	1	0	0	1	0	0	0	2
-1	1	0	0	0	1	0	0	4
2	-2	2	-1	0	0	-1	0	4
-2	2	1	1	0	0	0	-1	6

Pivot the primal slack & dual surplus variables into the basis:

1	2	3	4	5	6	7	8	b
2	1	0	0	1	0	0	0	2
-1	1	0	0	0	1	0	0	4
-2	2	-2	1	0	0	1	0	-4
2	-2	-1	-1	0	0	0	1	-6

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Not feasible!
The dual surplus variables are negative.

TABLEAU with artificial variable included

1	2	3	4	5	6	7	8	a	b
2	1	0	0	1	0	0	0	0	2
-1	1	0	0	0	1	0	0	0	4
-2	2	-2	1	0	0	1	0	-1	-4
2	-2	-1	-1	0	0	0	1	-1	-6

** ** ← *artificial variable*

The artificial variable has coefficient zero in feasible rows, and −1 in rows with infeasibility.

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Pivot the artificial variable into the basis, with the pivot in the row with maximum infeasibility

1	2	3	4	5	6	7	8	a	b
2	1	0	0	1	0	0	0	0	2
-1	1	0	0	0	1	0	0	0	4
-2	2	-2	1	0	0	1	0	-1	-4
2	-2	-1	-1	0	0	0	1	(-1)	-6

** ** ←

Artificial variable (a) enters the basis,
    replacing variable 8, whose complement is variable 2

1	2	3	4	5	6	7	8	a	b
2	1	0	0	1	0	0	0	0	2
-1	1	0	0	0	1	0	0	0	4
-4	4	-1	2	0	0	1	-1	0	2
-2	2	1	1	0	0	0	-1	1	6

** * *

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1 2	3 4	5 6	7 8	a	b
2 1	0 0	1 0	0 0	0	2
⁻1 1	0 0	0 1	0 0	0	4
⁻4 (4)	⁻1 2	0 0	1 ⁻1	0	2
-2 2	1 1	0 0	0 ⁻1	1	6

Entering: 2, Leaving: 7 (Pivot in row 3)

1 2	3	4	5 6	7	8	a	b
3 0	0.25	⁻0.5	1 0	⁻0.25	0.25	0	1.5
0 0	0.25	⁻0.5	0 1	⁻0.25	0.25	0	3.5
⁻1 1	⁻0.25	0.5	0 0	0.25	⁻0.25	0	0.5
0 0	1.5	0	0 0	⁻0.5	⁻0.5	1	5

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1 2	3	4	5 6	7	8	a	b
(3) 0	0.25	⁻0.5	1 0	⁻0.25	0.25	0	1.5
0 0	0.25	⁻0.5	0 1	⁻0.25	0.25	0	3.5
⁻1 1	⁻0.25	0.5	0 0	0.25	⁻0.25	0	0.5
0 0	1.5	0	0 0	⁻0.5	⁻0.5	1	5

Entering: 1, Leaving: 5 (Pivot in row 1)

1 2	3	4	5 6	7	8	a	b
1 0	0.083	⁻0.166	0.333 0	⁻0.083	0.083	0	0.5
0 0	0.25	⁻0.5	0 1	⁻0.25	0.25	0	3.5
0 1	⁻0.166	0.333	0.333 0	0.166	⁻0.166	0	1
0 0	1.5	0	0 0	⁻0.5	⁻0.5	1	5

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1	2	3	4	5	6	7	8	a	b
1	0	0.083	⁻0.166	0.333	0	⁻0.083	0.083	0	0.5
0	0	0.25	⁻0.5	0	1	⁻0.25	0.25	0	3.5
0	1	⁻0.166	0.333	0.333	0	0.166	⁻0.166	0	1
0	0	(1.5)	0	0	0	⁻0.5	⁻0.5	1	5

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Entering: 3, Leaving: 9 (Pivot in row 4)

1	2	3	4	5	6	7	8	a	b
1	0	0	⁻0.166	0.333	0	⁻0.055	0.111	⁻0.055	0.222
0	0	0	⁻0.5	0	1	⁻0.166	0.333	⁻0.166	2.666
0	1	0	0.333	0.333	0	0.111	⁻0.222	0.111	1.555
0	0	1	0	0	0	⁻0.333	⁻0.333	0.666	3.333

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*The artificial variable
has left the basis!*

OPTIMAL SOLUTION

```
Primal Variables: x = 0.222222 1.55556
        Slack:     y = 0 2.66667
Dual Variables:    u = 3.33333 0
        Surplus:   v = 0 0
Objective Function:  ⁻8.44444
```