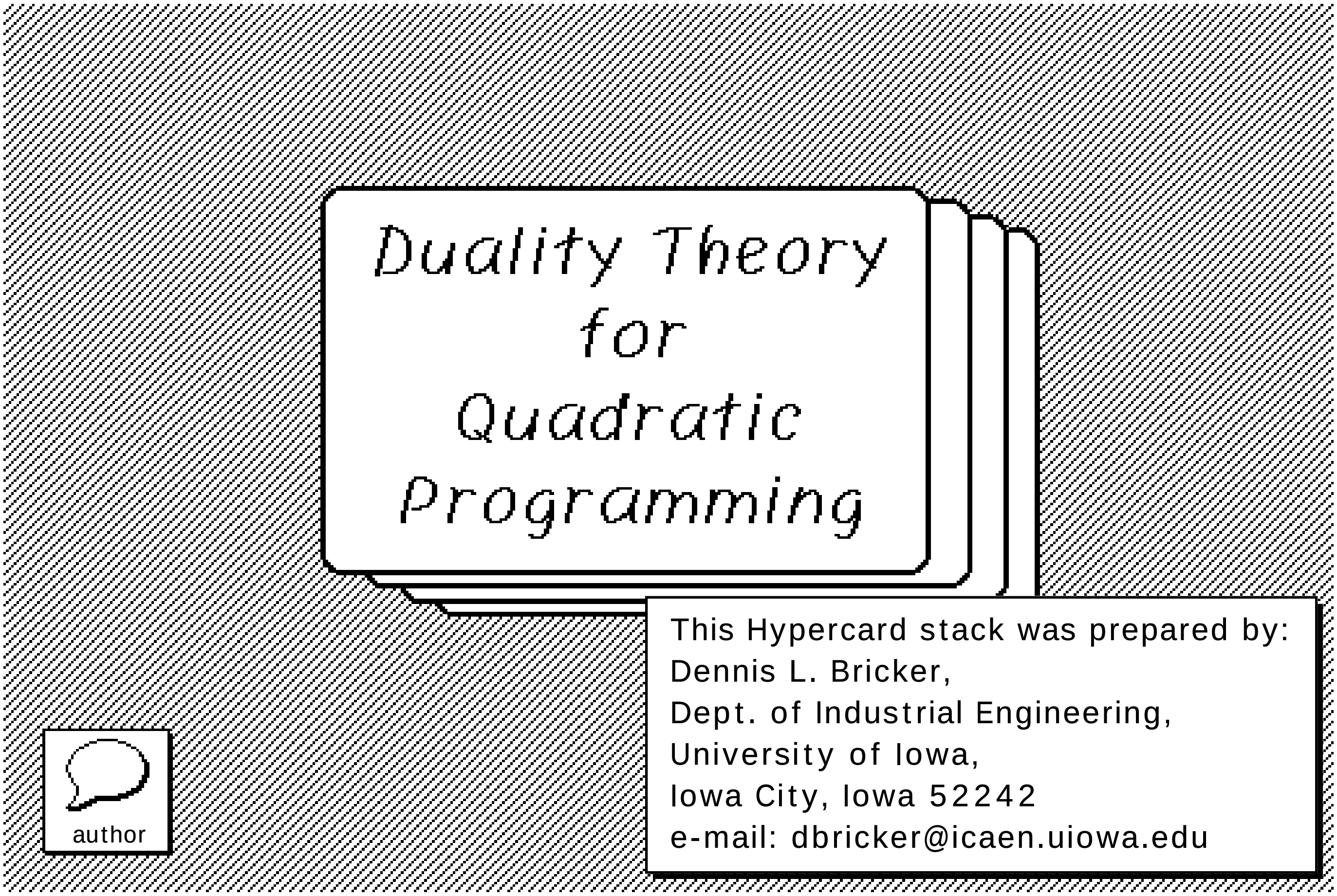
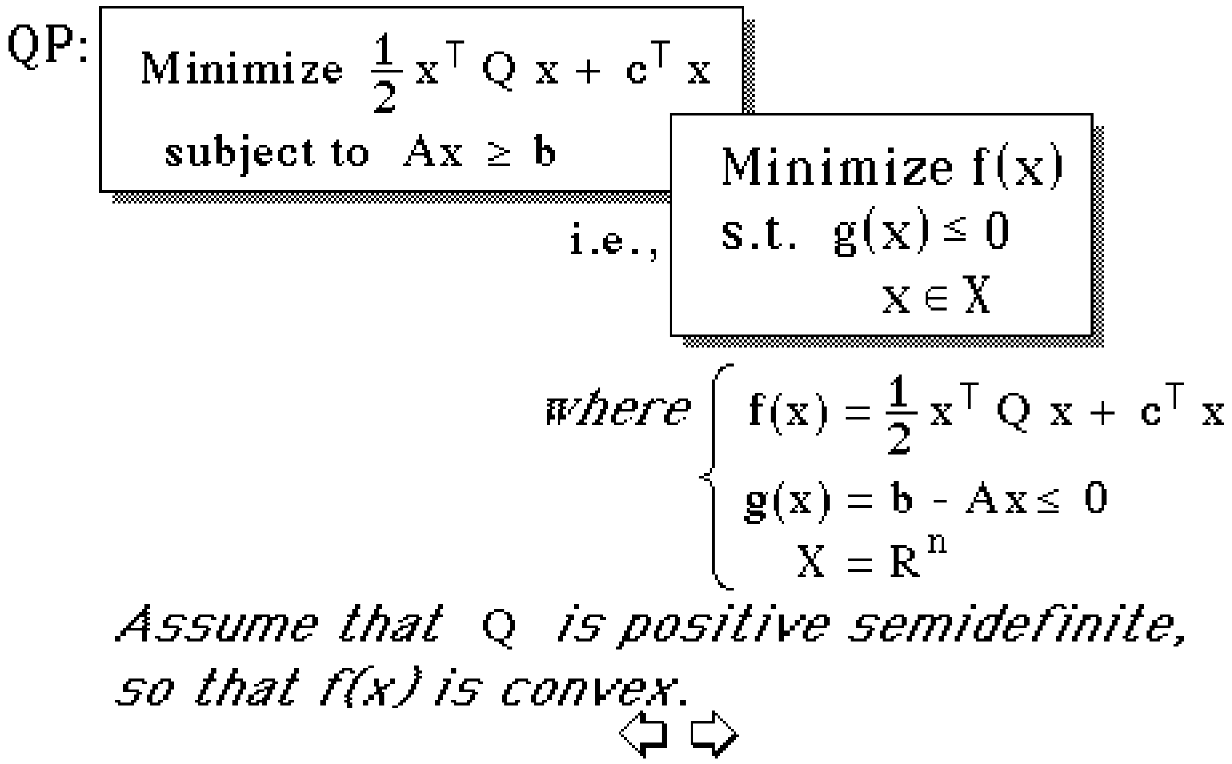


Duality Theory for Quadratic Programming

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu



QP:

$$\begin{aligned} &\text{Minimize } \frac{1}{2} x^{\top} Q\, x + c^{\top} x \\ &\text{subject to } Ax \geq b \end{aligned}$$

i.e.,

$$\begin{aligned} &\text{Minimize } f(x) \\ &\text{s.t. } g(x) \leq 0 \\ &\qquad x \in X \end{aligned}$$

where

$$\begin{cases} f(x) = \frac{1}{2} x^{\top} Q\, x + c^{\top} x \\ g(x) = b - Ax \leq 0 \\ X = R^{n} \end{cases}$$

Assume that Q *is positive semidefinite, so that f(x) is convex.*

⇦ ⇨

Lagrangian Function

$$\begin{aligned} L(x,\lambda) &= f(x) + \lambda^T g(x) \\ &= \frac{1}{2} x^T Q x + c^T x + \lambda^T (b - Ax) \end{aligned}$$

Dual Objective Function

$$\widehat{L}(\lambda) = \min_x \left\{ \frac{1}{2} x^T Q x + c^T x + \lambda^T (b - Ax) \right\}$$

For each value of λ, an unconstrained minimization of a convex quadratic function must be performed!

⇦ ⇨

Because of the convexity of the Lagrangian function, the optimal x must be a stationary point of the Lagrangian function:

$$\nabla_x L(\bar{x}(\lambda), \lambda) = 0 \Leftrightarrow \widehat{L}(\lambda) = L(\bar{x}(\lambda), \lambda)$$

i.e., for each λ, we must choose x to satisfy

$$\begin{aligned} \nabla_x L(x, \lambda) &= Qx + c - A^T\lambda = 0 \\ &\Rightarrow x^T\left(Qx + c - A^T\lambda\right) = x^T(0) \\ &\Rightarrow x^TQx + x^Tc - x^TA^T\lambda = 0 \\ &\Rightarrow \boxed{x^T Qx + c^Tx - \lambda^TA\ x = 0} \end{aligned}$$

⇦ ⇨

Dual Objective Function

$$\widehat{L}(\lambda) = \min_{x} \left\{ \frac{1}{2} x^T Q x + c^T x + \lambda^T (b - Ax) \right\}$$

$$= \frac{1}{2} x^T Q x + c^T x + \lambda^T (b - Ax)$$

where x is chosen to satisfy

$$x^T Qx + c^T x - \lambda^T A x = 0$$

⇦ ⇨

Dual Objective Function

$$\widehat{L}(\lambda) = \frac{1}{2} x^T Q x + c^T x + \lambda^T (b - Ax)$$

where $x^T Qx + c^T x - \lambda^T A x = 0$

$$= \lambda^T b - \frac{1}{2} x^T Q x + \underbrace{x^T Q x + c^T x - \lambda^T A x}_{= 0}$$

Therefore,

$\widehat{L}(\lambda) = \lambda^T b - \frac{1}{2} x^T Qx$ *where x must satisfy*

$$Qx + c - A^T \lambda = 0$$

⇦ ⇨

LAGRANGIAN DUAL OF QP

$$\underset{\lambda \geq 0}{\text{Maximize}}\ \widehat{L}(\lambda)$$

$$\text{Maximize}\ \lambda^{T} b - \frac{1}{2} x^{T} Q\, x$$

$$\text{subject to}\quad Qx + c - A^{T}\lambda = 0$$

$$\lambda \geq 0$$

That is, the Lagrangian dual of the quadratic programming problem QP is another quadratic programming problem with only nonnegativity constraints!

⇦ ⇨

If Q is positive definite, i.e., f(x) is strictly convex, then Q is nonsingular, and

$$Qx + c - A^{T}\lambda = 0$$

can be solved by inverting Q:

$$\boxed{\bar{x}(\lambda) = Q^{-1}\left[A^{T}\lambda - c\right]}$$

⇦ ⇨

$$\boxed{\bar{x}(\lambda) = Q^{-1}\left[A^T\lambda - c\right]}$$

This can be used to eliminate x from the statement of the Dual Problem:

$$\text{Maximize } \lambda^T b - \frac{1}{2} x^T Q\, x = b^T\lambda - \frac{1}{2}\left[Q^{-1}(A^T\lambda - c)\right]^T Q \left[Q^{-1}(A^T\lambda - c)\right]$$

subject to $\lambda \geq 0$

⇦ ⇨

So the dual objective, expressed in terms of λ, is

$$b^T\lambda - \frac{1}{2}\left[Q^{-1}(A^T\lambda - c)\right]^T Q \left[Q^{-1}(A^T\lambda - c)\right]$$

$$= b^T\lambda - \frac{1}{2}\left[(A^T\lambda - c)^T Q^{-1}(A^T\lambda - c)\right]$$

$$= b^T\lambda - \frac{1}{2}\left[\lambda^T A Q^{-1} A^T\lambda - 2\, c^T Q^{-1} A^T\lambda + c^T Q^{-1} c\right]$$

$$= \left[b^T + c^T Q^{-1} A^T\right]\lambda - \frac{1}{2}\lambda^T\left[A\, Q^{-1} A^T\right]\lambda - \frac{1}{2} c^T Q^{-1} c$$

⇦ ⇨

Thus the dual problem can be written as

$$\boxed{\begin{array}{c}\text{Maximize } \; e^T\lambda + \frac{1}{2}\lambda^T D\,\lambda \\ \text{subject to } \lambda \geq 0\end{array}} \quad -\frac{1}{2}c^TQ^{-1}c$$

constant

where

$$\begin{cases} e = b + AQ^{-1}c \\ D = -AQ^{-1}A^T \end{cases}$$

⇦ ⇨

Compare the sizes of the two problems:

PRIMAL:	DUAL:
n variables	m variables
m constraints (inequalities)	m constraints (nonnegativity)

It would appear that the Dual QP problem is more computationally attractive...

especially if the number of primal variables is more than the number of constraints!

⇦ ⇨

However,

Note that in QP we included included no explicit nonnegativity constraints... if $x \geq 0$ is to be included, we must include in the constraints
$$\left[\begin{array}{c} A \\ \hdashline I \end{array}\right] x \geq \left[\begin{array}{c} b \\ \hdashline 0 \end{array}\right]$$

This adds n primal constraints ⇒ # of dual variables will be m+n.

⇦ ⇨

EXAMPLE

$$\text{Minimize } \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - 2x_1 - 2x_2$$

$$\text{subject to } \begin{cases} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \end{cases}$$

that is,

$$\text{Minimize } \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ -2 \end{bmatrix}^T x$$

$$\text{subject to } \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

⇦ ⇨

To write the dual QP, we must compute

$$\begin{cases} e = b + AQ^{-1}c \\ D = -AQ^{-1}A^T \end{cases}$$

using

$$\begin{cases} Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \\ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

⇦ ⇨

$$D = -AQ^{-1}A^T = -\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

⇦ ⇨

$$\mathbf{e} = \mathbf{b} + \mathbf{A}\mathbf{Q}^{-1}\mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}$$

⇦ ⇨

Dual QP Problem

$$\text{Maximize} \quad \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}^{T} \lambda \; + \frac{1}{2}\lambda^{T} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \lambda$$

$$\text{subject to } \lambda \geq 0$$

$$\left\{ \begin{array}{l} \text{Maximize } \lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4 \\ \qquad - \frac{1}{2}\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2\right] + \lambda_1\lambda_3 + \lambda_2\lambda_4 \\ \text{subject to } \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0 \end{array} \right.$$

⇦ ⇨

After finding the optimal dual solution ,
we can compute the optimal primal solution:

$$x^*(\lambda^*) = Q^{-1}\left[A^T\lambda^* - c\right]$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\lambda^* - \begin{bmatrix}-2\\ -2\end{bmatrix}\right)$$

$$\Longrightarrow \begin{cases} x_1^* = -\lambda_1^* + \lambda_3^* + 2\\ x_2^* = -\lambda_2^* + \lambda_4^* + 2 \end{cases}$$