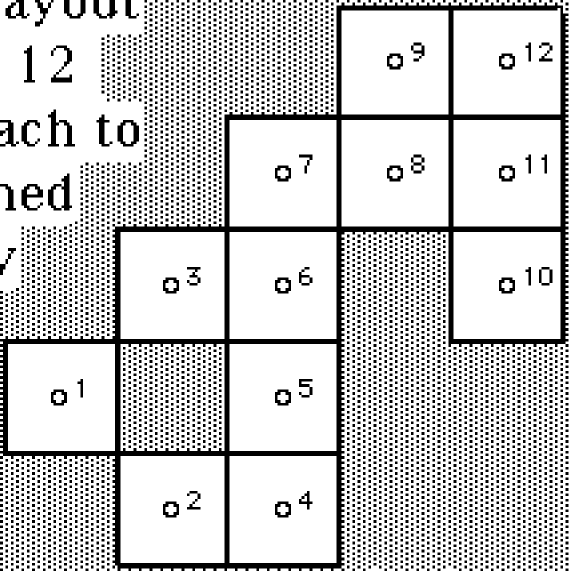
Quadratic Assignment Problem: a Simulated Annealing Algorithm

author

Example:
A floor layout contains 12 rooms, each to be assigned a facility



Define a binary decision variable for each combination of facility and location:

$$X_{ia} = \begin{cases} 1 & \text{if facility i is located at "a"} \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum_{i=1}^{n} X_{ia} = 1$ for each location $a=1, \ldots n$

(each location is to be assigned exactly one facility)

and $\sum_{a=1}^{n} X_{ia} = 1$ for each facility $i=1, \ldots n$

(each facility is to be assigned to a location)

Distance Matrix

to

from

D	1	2	3	4	5	6	7	8	9	10	11	12
1	0	2	2	3	2	3	4	5	6	5	6	7
2	2	0	2	1	2	3	4	5	6	5	6	7
3	2	2	0	3	2	1	2	3	4	3	4	5
4	3	1	3	0	1	2	3	4	5	4	5	6
5	2	2	2	1	0	1	2	3	4	3	4	5
6	3	3	1	2	1	0	1	2	3	2	3	4
7	4	4	2	3	2	1	0	1	2	3	2	3
8	5	5	3	4	3	2	1	0	1	2	1	2
9	6	6	4	5	4	3	2	1	0	3	2	1
10	5	5	3	4	3	2	3	2	3	0	1	2
11	6	6	4	5	4	3	2	1	2	1	0	1
12	7	7	5	6	5	4	3	2	1	2	1	0

(These are the "rectangular" distances between centers of the areas)

Interfacility Flow Matrix

F

to

from	A	B	C	D	E	F	G	H	I	J	K	L
A			3						4		2	
B									4			
C	3					5	1			1	1	4
D									1			
E								1				3
F			5							3		3
G			1								3	
H					1							
I	4	4		1						4		
J			1			3			4		1	
K	2		1				3			1		
L			4		3	3						

Density = 25.76%

If facility i is located at location "a", and facility j at "b", then the cost of the flow between this pair of facilities is assumed to be:

$$F_{ij} D_{ab}$$

Minimize

The optimization problem is to

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} F_{ij} D_{ab} X_{ia} X_{jb}$$

Note that the cost function is not linear, but QUADRATIC!

subject to

$$\sum_{i=1}^{n} X_{ia} = 1 \quad \text{for each location } a=1, \ldots n$$

(each location is to be assigned exactly one facility)

$$\sum_{a=1}^{n} X_{ia} = 1 \quad \text{for each facility } i=1, \ldots n$$

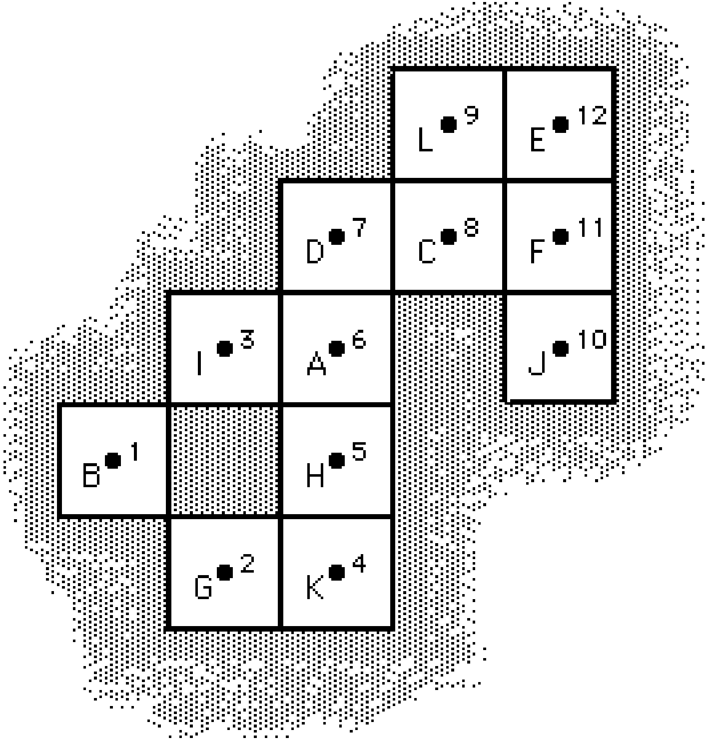
(each facility is to be assigned to a location)

$$X_{ia} \in \{ 1,0 \} \quad \text{for each } i=1, \ldots n \ \& \ a=1, \ldots n$$

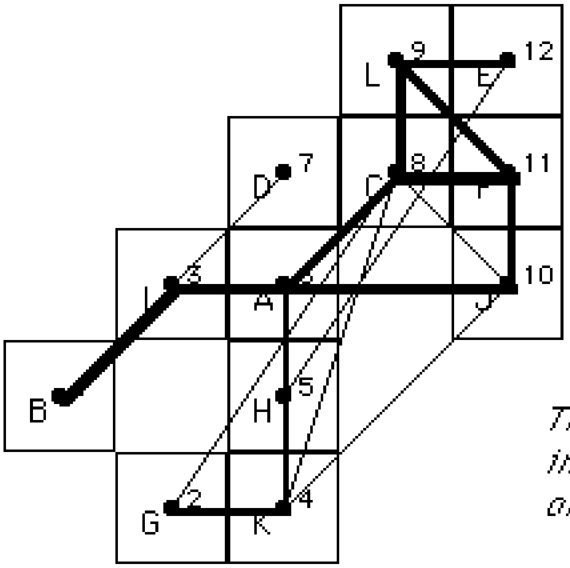
A heuristic solution:

Facility	Location
A	6
B	1
C	8
D	7
E	12
F	11
G	2
H	5
I	3
J	10
K	4
L	9

Cost: 160 sum of weighted distances



Material Flows

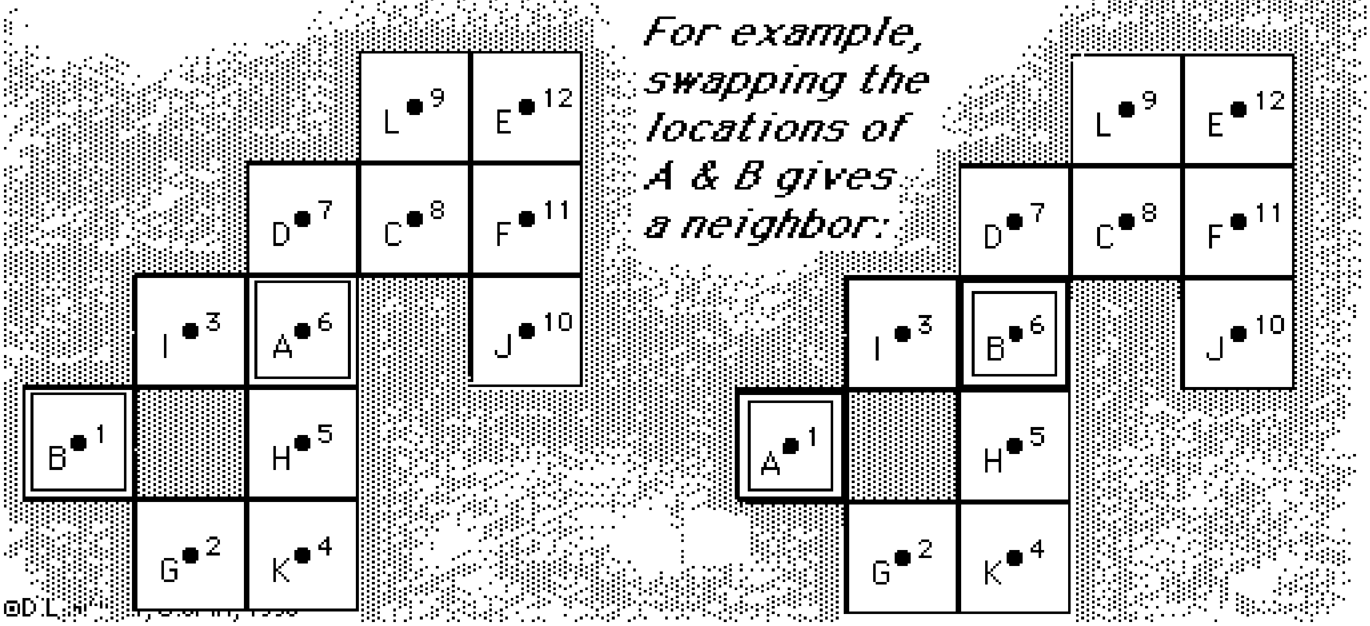


The thickness of the line indicates the magnitude of the flow

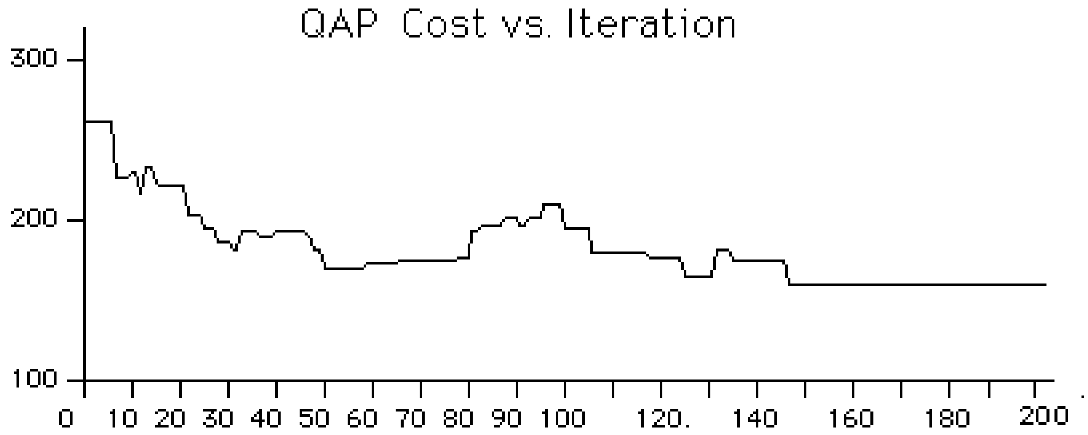
"Simulated Annealing"

- a heuristic search approach
- a move is made to any neighboring solution with equal or lower cost
- if the neighbor increases the cost by $\Delta > 0$, then the move is accepted with probability $P\{\text{accept } \Delta\} = e^{-\Delta/T}$ where T is the current "temperature" of the system
- the system is "cooled" according to some "cooling schedule"

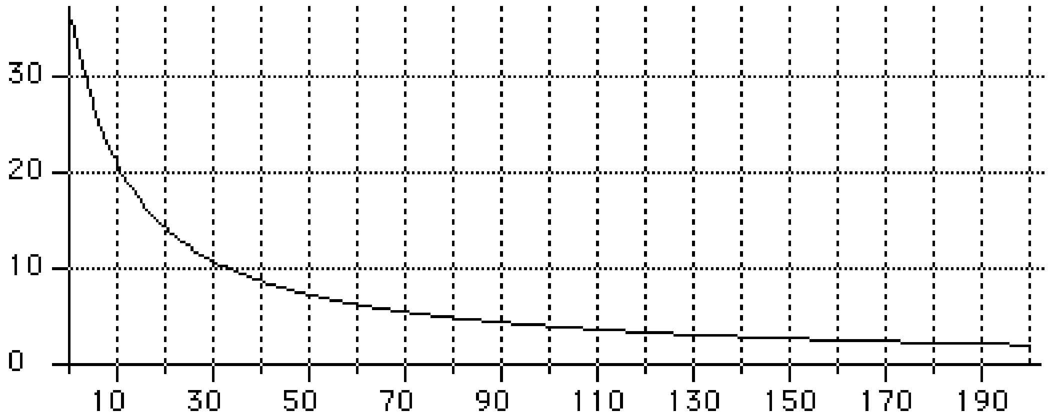
We will consider the neighbors of a solution to be those which result from a "swap" of the locations of 2 facilities


For example, swapping the locations of A & B gives a neighbor:

A typical simulated annealing result:

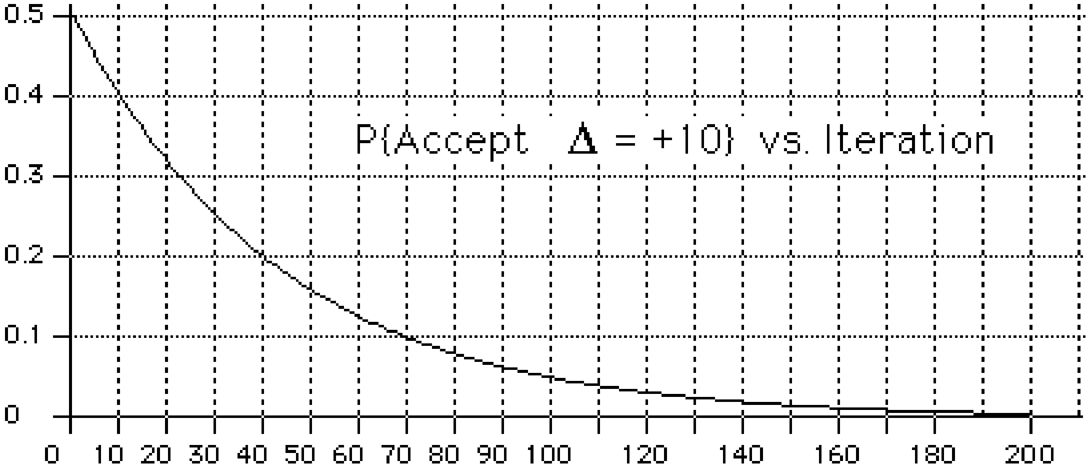


After each iteration, the temperature is reduced, according to a "cooling schedule"



$$\boxed{T_{i+1} = \frac{T_i}{1+\beta T_i}} \quad \text{where} \quad \beta = \frac{(T_0 - T_f)}{M\ T_0 T_f} \quad \& \begin{cases} T_0 = \text{initial temperature} \\ T_f = \text{final temperature} \\ M = \#\ \text{of iterations} \end{cases}$$

As the system "cools", the probability of accepting an increase (of 10) decreases:

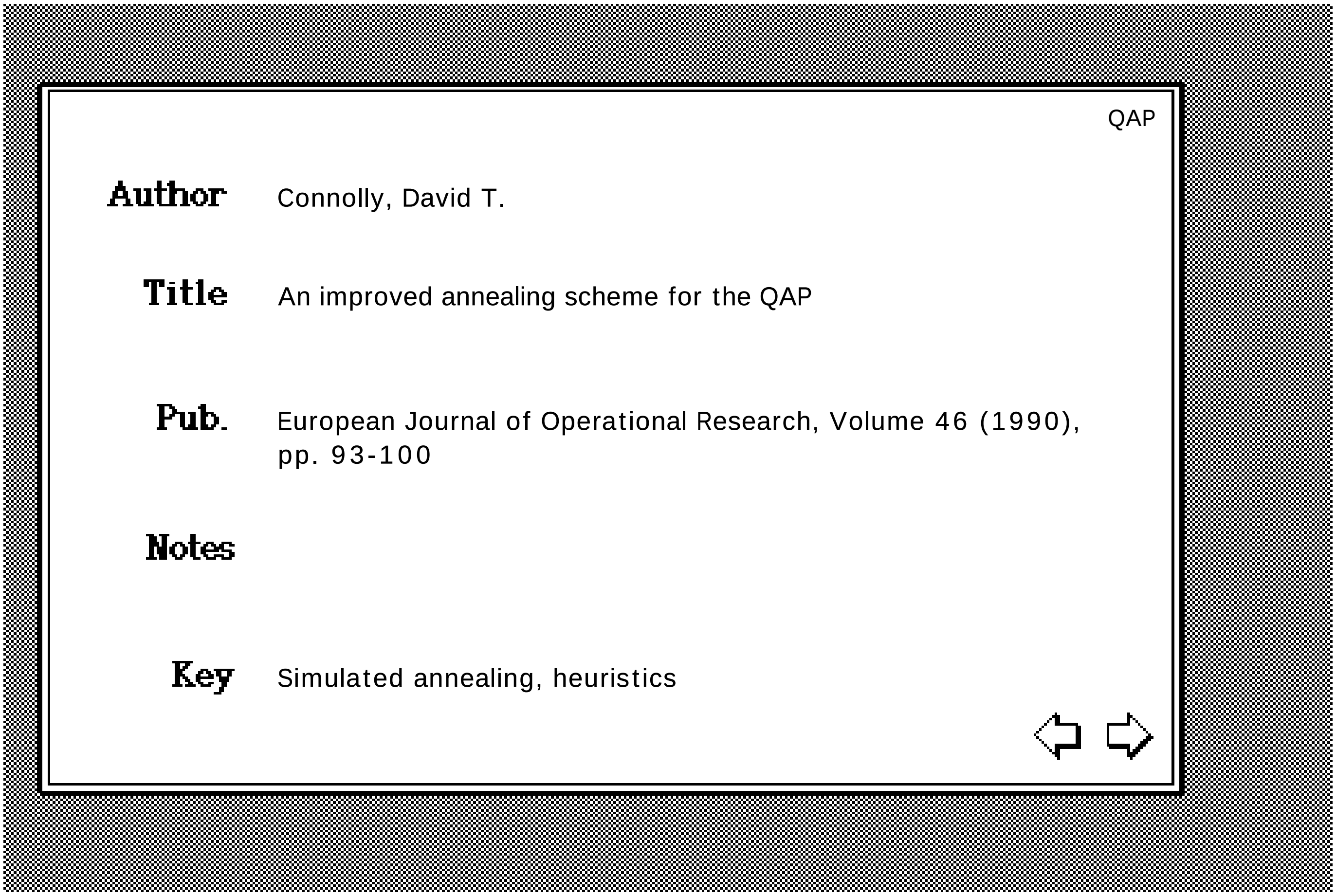


$$P\{\text{accept } \Delta\} = e^{-\Delta/T}$$

The first 15 iterations of a simulated annealing:

Iteration #	Temp	Z	Swap pair	Δ	P{accept}	Accept ?	
1	14.42695	262	(1↔ 3)	44	0.0474		
2	13.96311	262	(1↔ 9)	12	0.4234		
3	13.52816	262	(1↔11)	0	1.0000	Y	
4	13.11949	262	(2↔ 9)	18	0.2536		
5	12.73479	262	(3↔ 6)	28	0.1109		
6	12.37200	262	(3↔ 7)	¯34	1.0000	Y	↓
7	12.02932	228	(3↔10)	12	0.3688		
8	11.70510	228	(3↔11)	84	0.0008		
9	11.39791	228	(3↔12)	2	0.8391	Y	↑
10	11.10642	230	(4↔ 9)	0	1.0000	Y	
11	10.82948	230	(5↔ 8)	¯12	1.0000	Y	↓
12	10.56600	218	(5↔12)	16	0.2200	Y	↑
13	10.31505	234	(6↔10)	20	0.1439		
14	10.07574	234	(6↔12)	¯10	1.0000	Y	↓
15	9.84728	224	(7↔11)	¯2	1.0000	Y	↓

a swap which results in an increase is accepted!



QAP

Author Connolly, David T.

Title An improved annealing scheme for the QAP

Pub. European Journal of Operational Research, Volume 46 (1990), pp. 93-100

Notes

Key Simulated annealing, heuristics