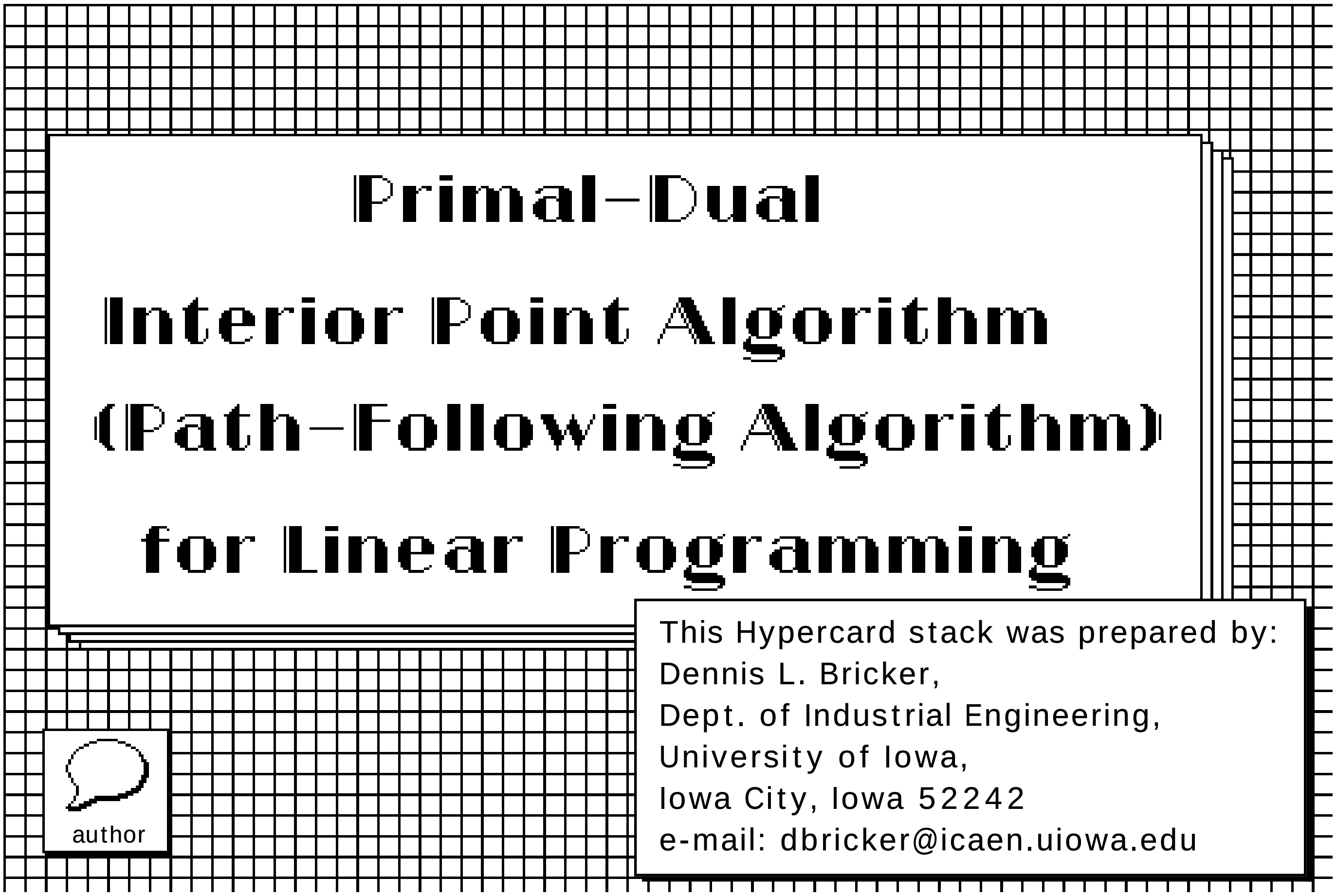

# Primal-Dual
# Interior Point Algorithm (Path-Following Algorithm) for Linear Programming

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Consider the primal/dual pair of LPs:

**Primal**

Minimize $c^t x$
subject to $Ax = b$
$x \geq 0$

**Dual**

Maximize $y\,b$
subject to $yA \leq c^t$

i.e.,

Maximize $b^t y$
subject to $A^t y \leq c$

## Convert dual constraints to equalities:

**Primal**

Minimize $c^t x$
subject to $Ax = b$
$x \geq 0$

**Dual**

Maximize $b^t y$
subject to $A^t y + z = c^t$
$z \geq 0$

Use barrier functions to relax the non-negativity conditions:

**Primal**

Minimize $c\,x - \boldsymbol{\mu} \sum_{j=1}^{n} \ln(x_j)$
subject to
$A\,x = b$

as $x \longrightarrow 0$,
$-\boldsymbol{\mu} \ln(x) \longrightarrow \infty$

**Dual**

Maximize $b^t y + \boldsymbol{\mu} \sum_{j=1}^{n} \ln(z_j)$
subject to
$A^t y + z = c^t$

Use Lagrange multipliers to relax the equality constraints:

*Lagrangian Functions*

$$L_P(x, y) = c^t x - \boldsymbol{\mu} \sum_{j=1}^{n} \ln(x_j) + y^t(A x - b)$$

$$L_D(x,y,z) = b^t y + \boldsymbol{\mu} \sum_{j=1}^{n} \ln(x_j) - x^t (A^t y + z - c)$$

The optimality conditions may be written

$$\boxed{\frac{\partial L_P(x,y)}{\partial x} = 0, \quad \frac{\partial L_P(x,y)}{\partial y} = 0}$$

and

$$\boxed{\frac{\partial L_D(x,y,z)}{\partial x} = 0, \frac{\partial L_D(x,y,z)}{\partial y} = 0, \frac{\partial L_D(x,y,z)}{\partial z} = 0}$$

*These reduce to the following*

optimality conditions

$$\begin{array}{l} \left.\begin{array}{c} A\,x = b \\ A^t y + z = c \end{array}\right\} \textit{linear equations} \\ x_j\, z_j = \boldsymbol{\mu},\ j=1,2,\ \ldots n \;\leftarrow \textit{nonlinear equations} \end{array}$$

To solve the nonlinear system of equations, we might use the ***Newton-Raphson*** method:

Given an initial approximate solution $(x^0, y^0, z^0)$: an improved approximate solution is given by

$$\left\{\begin{array}{l} x^1 = x^0 + \boldsymbol{\delta}_x \\ y^1 = y^0 + \boldsymbol{\delta}_y \\ z^1 = z^0 + \boldsymbol{\delta}_z \end{array}\right.$$

where $\boldsymbol{\delta}_x$, $\boldsymbol{\delta}_y$, and $\boldsymbol{\delta}_z$ are found by solving a linear system.

## *Notation*

$$\mathbf{X} = \operatorname{diag}\{x_1, x_2, \ldots x_n\}$$

$$Z = \operatorname{diag}\{z_1, z_2, \ldots z_n\}$$

$$e = [\,1\,,\ 1\,,\ \ldots .\ 1\,]$$

Then the constraints

$$x_j\, z_j = \boldsymbol{\mu},\ j=1,2,\ \ldots n$$

may be written

$$\mathbf{X}\,\mathbf{Z}\,e = \ \boldsymbol{\mu}\,e$$

We wish to solve the ***nonlinear*** system

$$\begin{cases} A\,x - b = 0 \\ A^t\,y + z - c = 0 \\ \mathbf{X}\ \mathbf{Z}\ e - \boldsymbol{\mu}\ e = 0 \end{cases}$$

Newton-Raphson Method: given $(x^o, y^o, z^o)$, solve the ***linear*** system

$$\begin{cases} A\ \delta_x & & = -\left[A x^0 - b\right] \\ & A^t\ \delta_y + \ \delta_z & = -\left[A^t\ y^0 + z^0 - c\right] \\ Z\ \delta_x & + \mathbf{X}\ \delta_z & = -\left[\mathbf{X}\ \mathbf{Z}\ e - \boldsymbol{\mu}\ e\right] \end{cases}$$

That is, solve

$$\textit{Jacobian matrix} \rightarrow \begin{bmatrix} A & 0 & 0 \\ 0 & A^t & \mathbf{I} \\ Z & 0 & \mathbf{X} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} = \begin{bmatrix} d_P \\ -d_D \\ \boldsymbol{\mu} e - \mathbf{X} Z e \end{bmatrix}$$

where $d_P = b - Ax^0$ ← *primal infeasibility*

$d_D = A^t y^0 + z^0 - c$ ← *dual infeasibility*

and then compute the improved approximation

$$\begin{cases} x^1 = x^0 + \boldsymbol{\delta}_x \\ y^1 = y^0 + \boldsymbol{\delta}_y \\ z^1 = z^0 + \boldsymbol{\delta}_z \end{cases}$$

Solving the linear system:

$$\delta_x = Z^{-1}\left[\boldsymbol{\mu} e - \mathbf{X} Z e - \mathbf{X}\, \delta_z\right]$$

$$\delta_z = -d_D - A^t \delta_y$$

$$\Longrightarrow \left[A Z^{-1} \mathbf{X} A^t\right] \delta_y = b - \boldsymbol{\mu} A Z^{-1} e - A Z^{-1} \mathbf{X}\, d_D$$

or $$\delta_y = \left[A Z^{-1} \mathbf{X} A^t\right]^{-1} \left(b - \boldsymbol{\mu} A Z^{-1} e - A Z^{-1} \mathbf{X} d_D\right)$$

Computing

$$\delta_y = \left[A\, Z^{-1} \mathbf{X}\, A^t\right]^{-1} \left(b - \mu A\, Z^{-1} e - A Z^{-1} \mathbf{X} d_D\right)$$

by using matrix inversion is computationally costly for large problems...
other methods for solving the linear system for $\delta_y$ are preferred.

After computing the step $(\delta_x, \delta_y, \delta_z)$,

$$\begin{cases} x^1 = x^0 + \delta_x \\ y^1 = y^0 + \delta_y \\ z^1 = z^0 + \delta_z \end{cases}$$

An alternative would be to go (almost) as far as possible in the $x$ direction and the $(y,z)$ direction:

$$\begin{cases} x^1 = x^0 + \alpha_P\, \delta_x \\ y^1 = y^0 + \alpha_D\, \delta_y \\ z^1 = z^0 + \alpha_D\, \delta_z \end{cases}$$

for stepsizes $\alpha_P$ and $\alpha_D$, respectively.

$$\alpha_P = \tau \min_j \left\{ \frac{-x_j^0}{\delta_{xj}} : \delta_{xj} < \mathbf{0} \right\}$$

$$\alpha_D = \tau \min_j \left\{ \frac{-z_j^0}{\delta_{zj}} : \delta_{zj} < \mathbf{0} \right\}$$

for $0 < \tau < 1$ *e.g.,* $\tau = 0{,}995$
( $\tau = 1$ will result in one of the x and z variables reaching zero!)

Generally, only one Newton-Raphson step is used, so that the nonlinear system is only approximately solved.

This completes one iteration. As $\mu \longrightarrow 0$, the values of x,y, and z will converge to the optimal primal and dual solutions.

The path followed by (x,y,z) is referred to as
the ***central path***
and the algorithm as
a ***path-following*** algorithm.

Reduction of $\mu$:

$$\mu = \frac{c^t x^1 - b^t y^1}{\theta(n)}$$

suggested value of parameter $\theta$:

$$\theta(n) = \begin{cases} n^2 \text{ if } n \leq 5{,}000 \\ n\sqrt{n} \text{ if } n > 5{,}000 \end{cases}$$

Termination criterion:

$$\frac{c^t x^k - b^t y^k}{1 + \left|b^t y^k\right|} < \varepsilon$$

The number of iterations required is rather insensitive to the size n of the problem, and is usually between 20 and 80 for most problems.