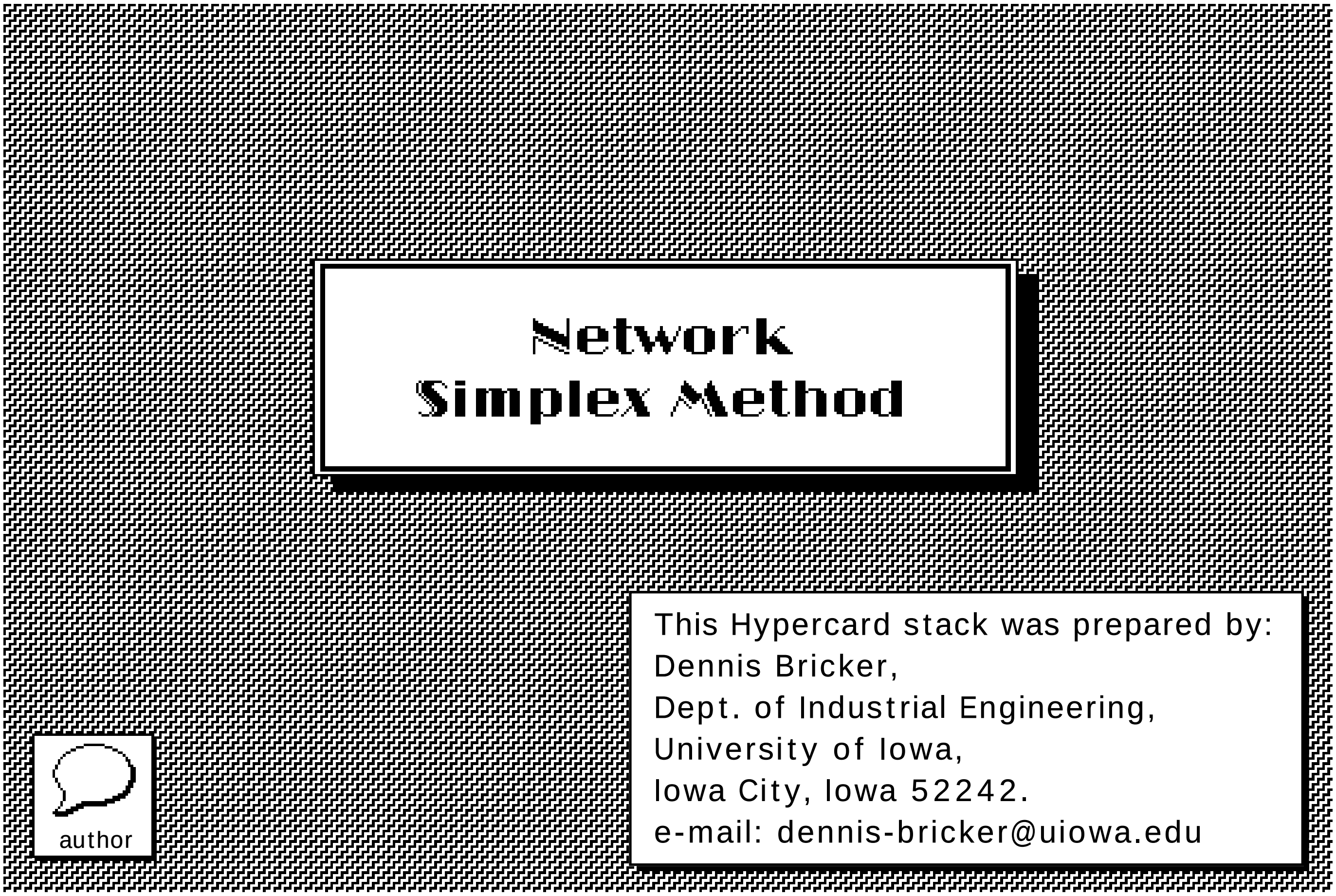


Algorithms for Min-Cost Network Flow Problem

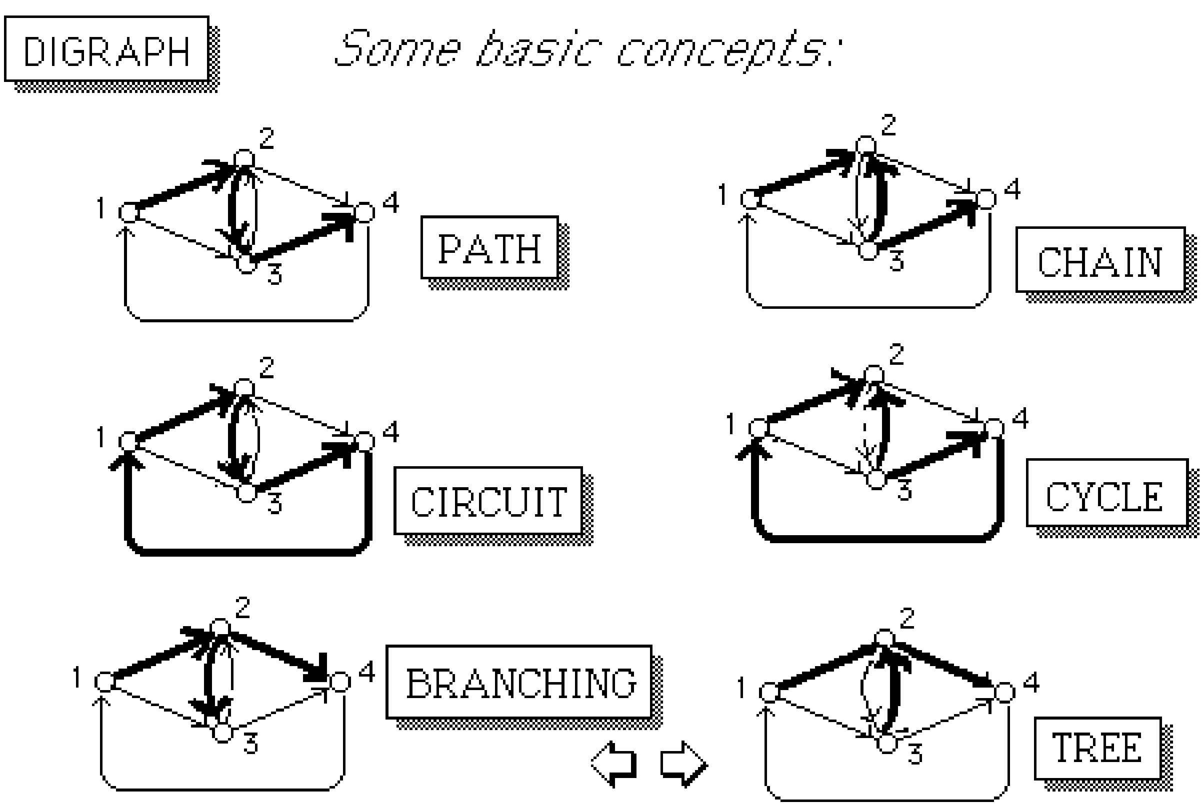
- Primal Simplex Method
- Out-of-Kilter (primal-dual) Method

  advantage: can easily re-optimize when costs remain same but supply/demand changes.
  (assumes circulation model of flow.)

We will apply the primal simplex method to the minimum-cost network flow problem, but (as was the case with the transportation problem) without pivoting in the full tableau.

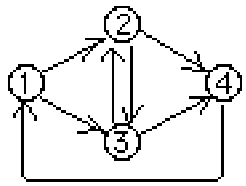
Questions to consider:

- How is basis matrix represented?
- How is simplex multiplier vector computed?
- How is change of basis accomplished?

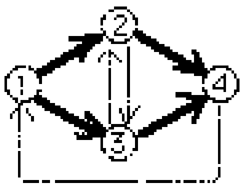


The columns of the node-arc incidence matrix corresponding to the arcs of a ***cycle*** are linearly ***dependent***.

Example



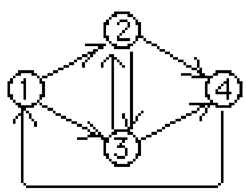
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$



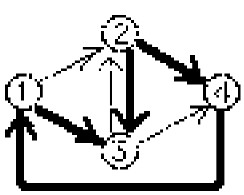
$$+\underset{(1,2)}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}} + \underset{(2,4)}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}} - \underset{(3,4)}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}} - \underset{(1,3)}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The sum of the columns is the zero vector!

Example



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

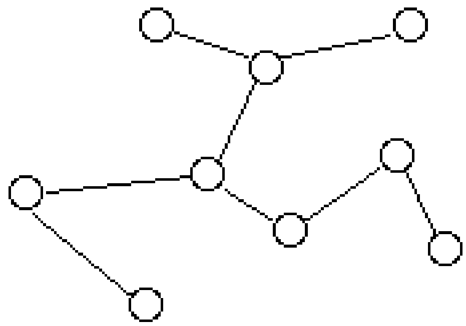


$$+\underset{(1,3)}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}} - \underset{(2,3)}{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}} + \underset{(2,4)}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}} + \underset{(4,1)}{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Send a unit of flow around the cycle... the coefficient of the column will be +1 if the flow is in direction of arc, and −1 if in opposite direction!

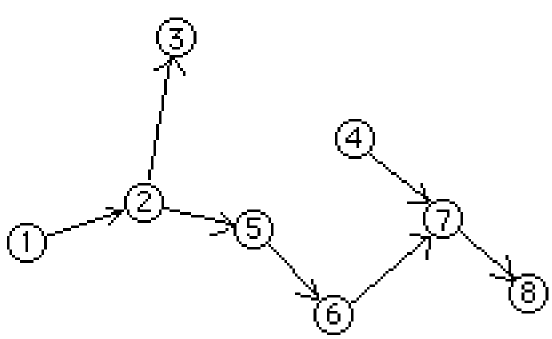
Theorem

A tree containing **m** nodes contains **m−1** arcs



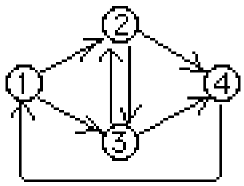
Removing a terminal node and its incident arc leaves a tree.
m−1 nodes can be so removed (along with m−1 arcs!), leaving finally a single node but no arc.

The Node-Arc incidence matrix of a tree is, after rearranging rows &/or columns, ***Lower Triangular***

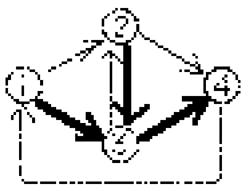


All non-zero values lie on or below the diagonal!

	(1,2)	(2,3)	(4,7)	(2,5)	(5,6)	(6,7)	(7,8)
1	**1**	0	0	0	0	0	0
3	0	**−1**	0	0	0	0	0
4	0	0	**1**	0	0	0	0
2	−1	1	0	**1**	0	0	0
5	0	0	0	−1	**1**	0	0
6	0	0	0	0	−1	**1**	0
7	0	0	−1	0	0	−1	**1**
8	0	0	0	0	0	0	−1

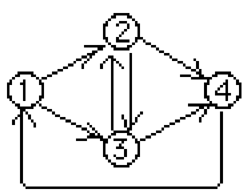


$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

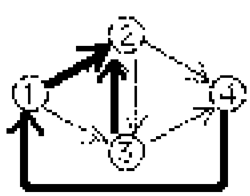


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \\ \hdashline 0 & 0 & -1 \end{bmatrix}$$

Rank of the matrix is 3



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$



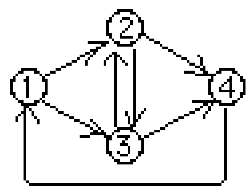
$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} 3 \\ 2 \\ 1 \\ 4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \\ \hdashline 0 & 0 & 1 \end{bmatrix}$$

Rearrange rows & columns... matrix is lower triangular!

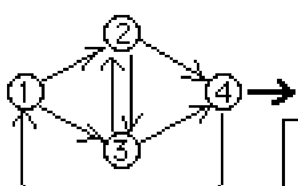
Rank is 3

Recall: rank of node-arc incidence matrix of a network is < m (# nodes)
rank of node-arc incidence matrix of a spanning tree is m−1



1	1	0	0	0	0	−1
−1	0	1	1	−1	0	0
0	−1	−1	0	1	1	0
0	0	0	−1	0	−1	1

rows are linearly dependent, so rank < 4



"root arc"

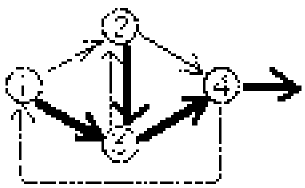
1	1	0	0	0	0	−1	**0**
−1	0	1	1	−1	0	0	**0**
0	−1	−1	0	1	1	0	**0**
0	0	0	−1	0	−1	1	**1**

column for the artificial variable

Inserting an artificial variable in some row makes the rank = m

The artificial variable corresponds to an arc which leaves a node but enters no other node!

Any basis matrix of the node-arc incidence matrix is the node-arc incidence matrix of a spanning tree, plus the column for the artificial variable.

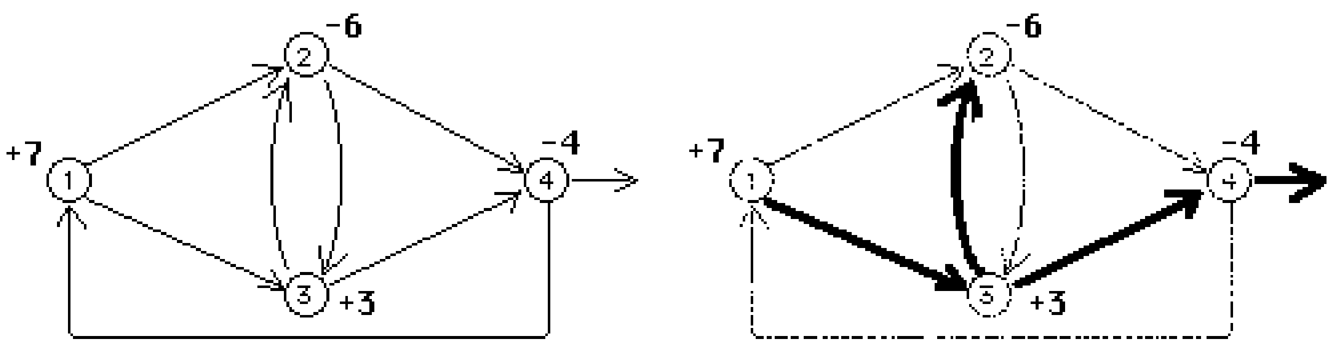


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ \hdashline 0 & 0 & -1 & 1 \end{bmatrix}$$

The lower-triangular basis matrix means that $A^B x_B = b$ *can be solved by* **forward substitution** *to get* $x_B = \left[A^B\right]^{-1} b$
We will actually implement this without explicitly writing the system of equations.

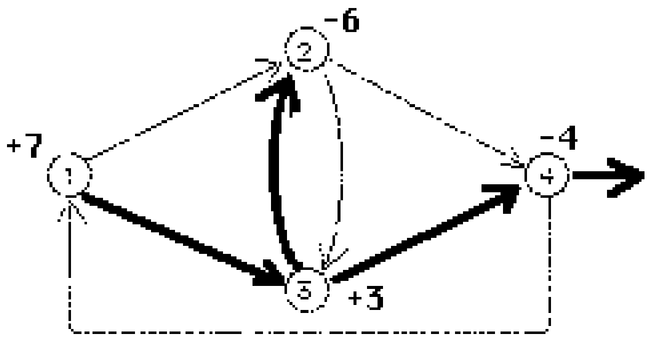
Computing the Basic Solution

(flow in the "rooted" spanning tree)



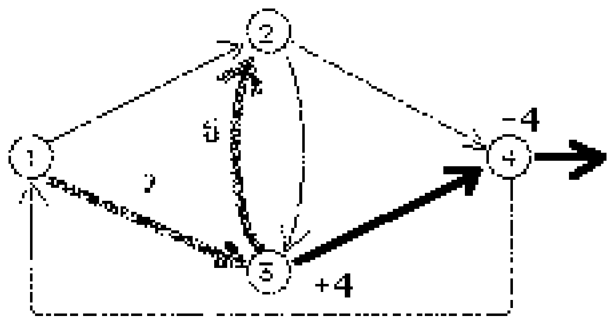
Beginning at the ends of the tree, assign flows until you reach the root.

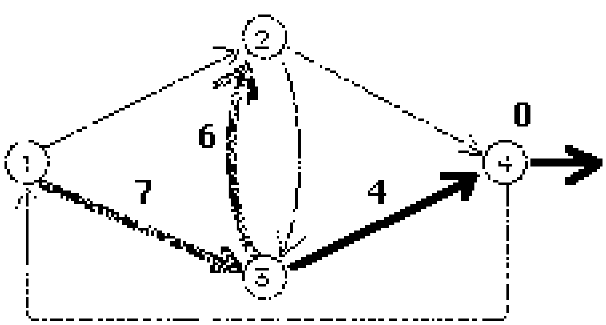
basic flows shown in bold



$X_{13} = 7$ and $X_{32} = 6$

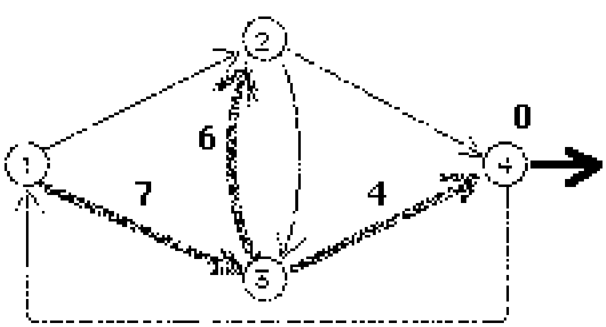
Update "supply" at node 3 and "trim" arcs (1,3) and (3,2) from the tree.
Node 3 is now an end.





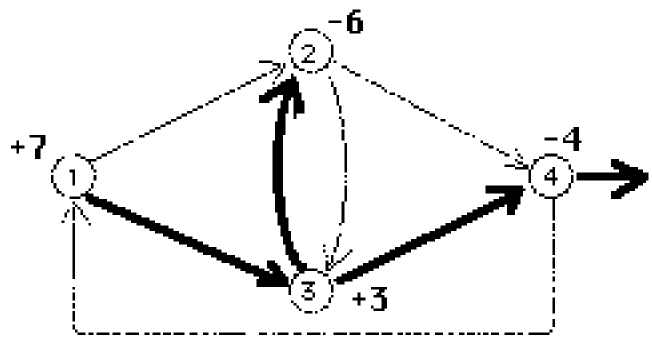
$X_{34} = 4$.

Trim (3,4), leaving node 4 as an end.



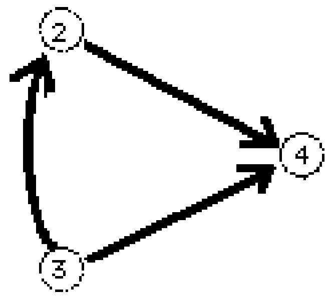
Flow in root node is zero.

Expressing a nonbasic arc as a combination of basic arcs

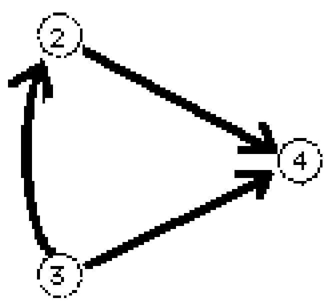


To write arc (2,4) as a combination of basic arcs:

(necessary to make a change of basis, i.e., pivot)



Inserting arc (2,4) into the spanning tree creates a cycle



The columns corresponding to the arcs of a cycle are linearly dependent

$$+\overset{(2,4)}{\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}}-\overset{(3,4)}{\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}}+\overset{(3,2)}{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Form the combination by going through the cycle in same direction as arc added, adding arcs oriented opposite & subtracting arcs oriented in same direction

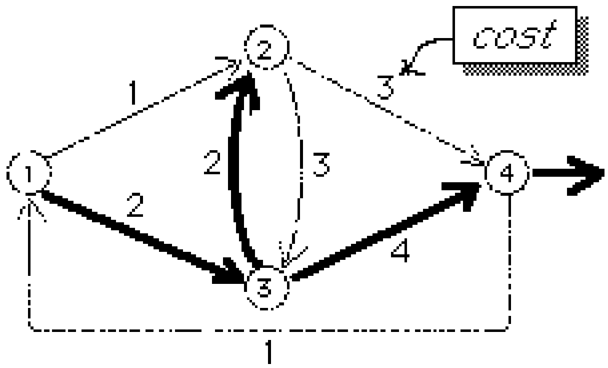
That is, $\overset{(2,4)}{\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}}=\overset{(3,4)}{\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}}-\overset{(3,2)}{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}}$

in order to select nonbasic variable to enter the basis

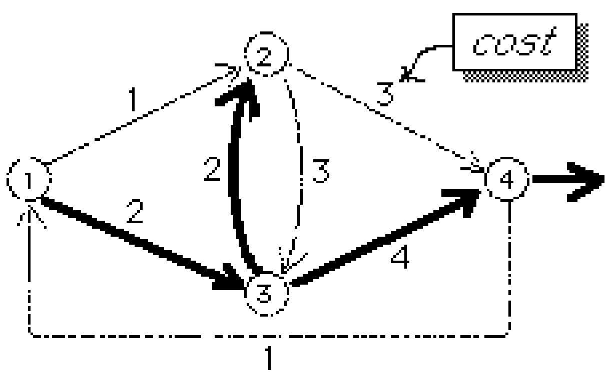
Pricing Nonbasic Arcs

Reduced cost of (i,j) is $C_{ij} - Z_{ij}$, where

Z_{ij} = cost of combination of basic arcs which is equivalent to nonbasic arc (i,j)



What is the reduced cost of nonbasic arc (2,4)?



Arc (2,4) = Arc (3,4) – Arc (3,2)

so

$$Z_{24} = C_{34} - C_{32} = 4 - 2 = 2$$

and reduced cost is $C_{24} - Z_{24} = 3 - 2 = 1 > 0$

Arc (2,4) shouldn't enter the basis!

Pricing Nonbasic Arcs

(An easier approach!)

Doesn't require expressing nonbasic flow as combination of basic flows

Reduced cost of arc (i,j) = $C_{ij} - w\,A^{ij}$

$= C_{ij} - (w_i - w_j)$

where w is vector of Simplex Multipliers

and A^{ij} is the column of the node-arc incidence matrix for arc (i,j)

How can we compute the Simplex Multipliers?

Computing Simplex Multipliers

$w = C_B (A^B)^{-1}$

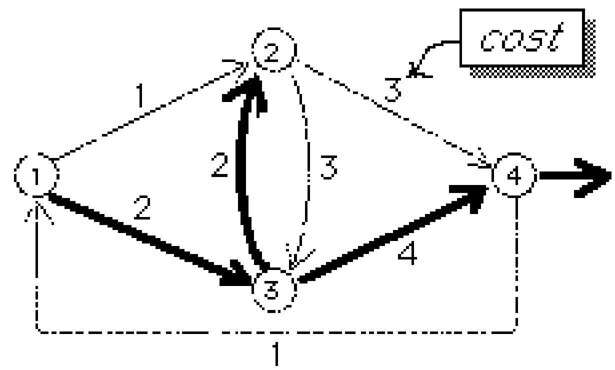
i.e., $w A^B = C_B$

or $w_i - w_j = C_{ij}$ for each basic arc (i,j)

Because of the fact that the basis matrix is (possibly after rearranging rows &/or columns) lower triangular, these equations are simple to solve for w.

$$wA^B = C_B \quad \Rightarrow$$

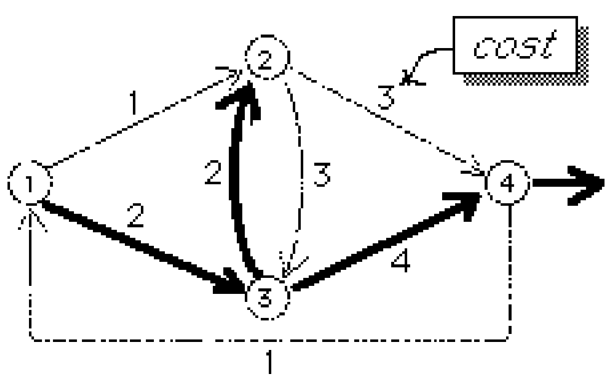
$$[w_1 \;\; w_2 \;\; w_3 \;\; w_4]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = [2 \;\; 2 \;\; 4 \;\; 0] \leftarrow C_B$$



Solve by "back-substitution"

$$\Rightarrow \begin{cases} w_1 - w_3 = 2 \\ -w_2 + w_3 = 2 \\ w_3 - w_4 = 4 \\ w_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} w_4 = 0 \\ w_3 = 4 + w_4 = 4 \\ w_2 = -2 + w_3 = 2 \\ w_1 = 2 + w_3 = 6 \end{cases}$$



$$\begin{cases} w_1 = 6 \\ w_2 = 2 \\ w_3 = 4 \\ w_4 = 0 \end{cases}$$

Reduced Costs:

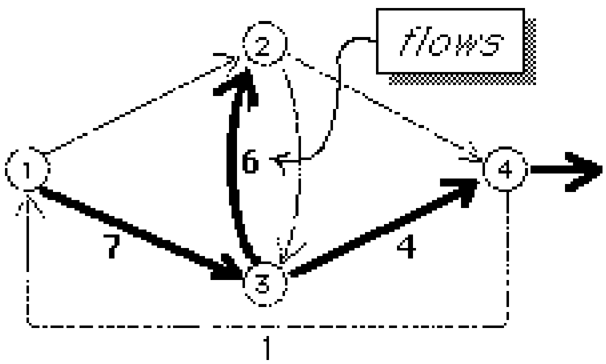
arc (1,2): $1 - (6-2) = -3$ ← *negative reduced cost indicates arc to enter the basis*

arc (2,3): $3 - (2-4) = +5$

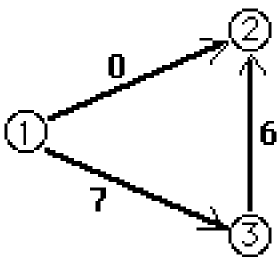
arc (2,4): $3 - (2-0) = +1$

arc (4,1): $1 - (0-6) = +7$

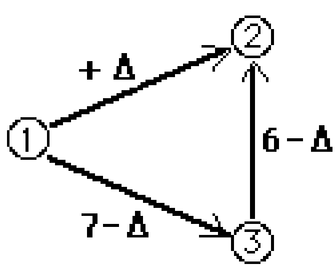
Choosing the Arc to Leave the Basis



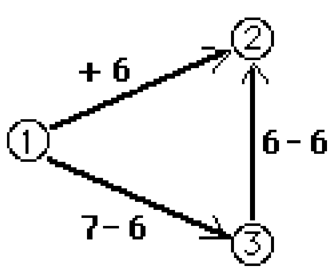
Suppose that arc (1,2) is to enter the basis, i.e., the tree.



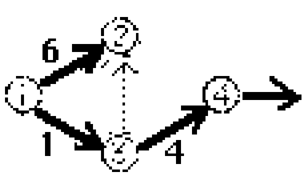
Identify the cycle created by inserting arc (1,2)



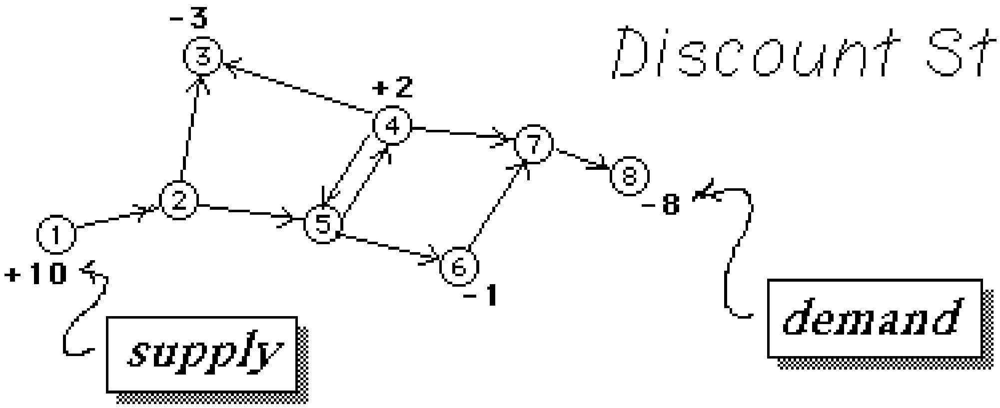
Send an amount of flow Δ around this cycle in direction of (1,2).
(Sending flow against direction of an arc will ***decrease*** *flow on the arc.)*



Increase Δ until the flow in some arc in the cycle drops to zero. Remove this arc from the tree.

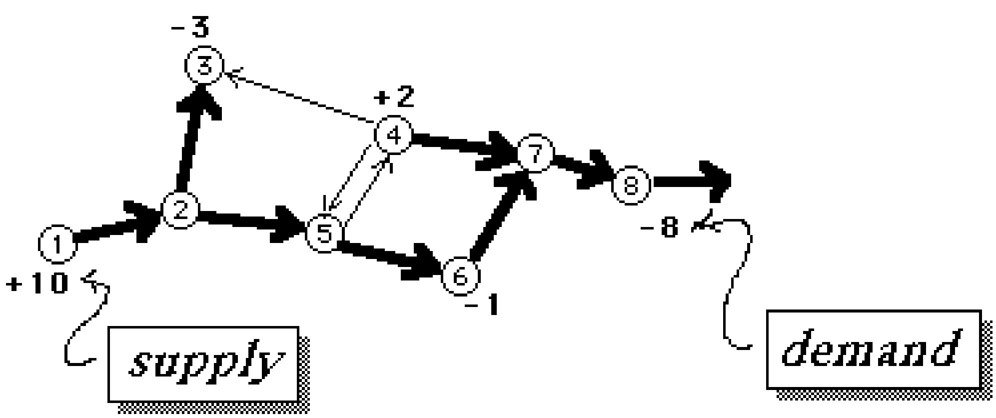


Rock-Bottom Discount Stores



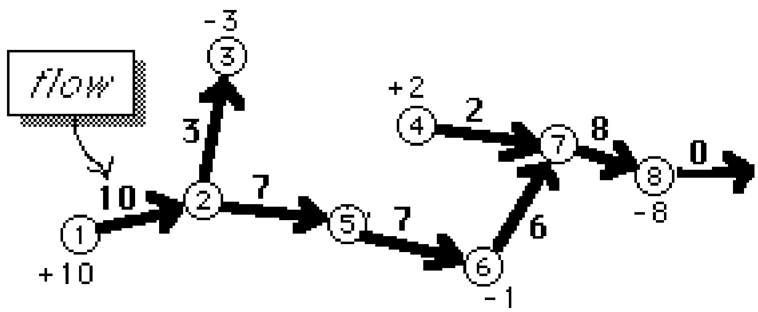
Example

An initial basis (rooted spanning tree):



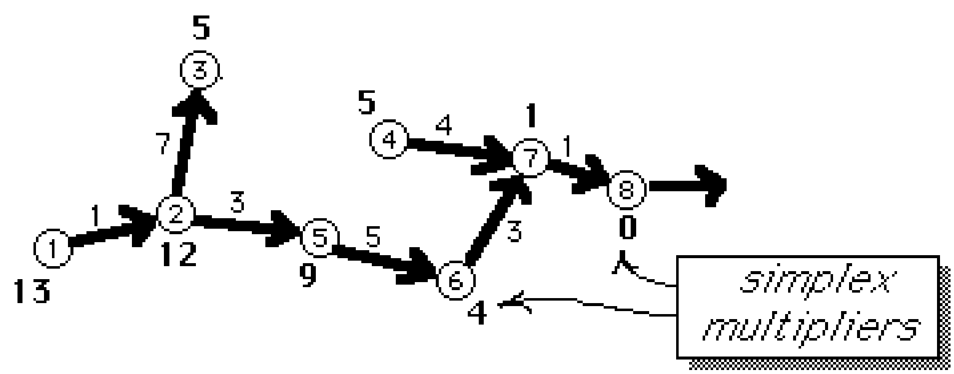
basic flows shown in bold

Basic solution:



basic flows shown in bold

Computing the Simplex Multipliers



$w_8 = 0$

$w_7 = 1 + w_8$

$w_4 = 4 + w_7$

$w_6 = 3 + w_7$

$w_5 = 5 + w_6$

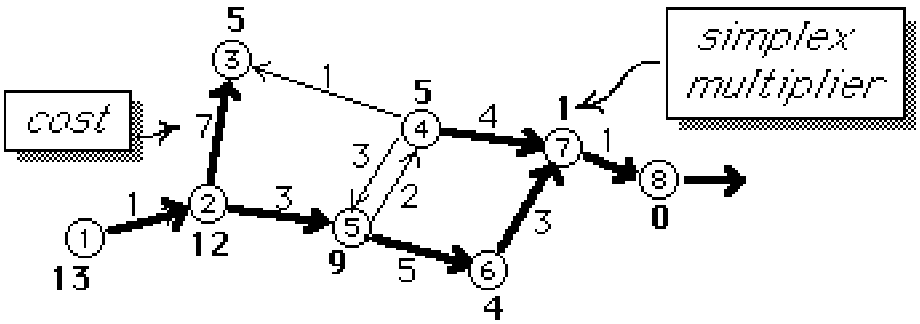
etc.

For each basic arc (i,j), $W_i - W_j = C_{ij}$

Start with "root", assign arbitrary value 0, and work your way to the ends of the branches.

$$W_i = C_{ij} + W_j$$

Computing Reduced Costs $\overline{C}_{ij} = C_{ij} - (W_i - W_j)$

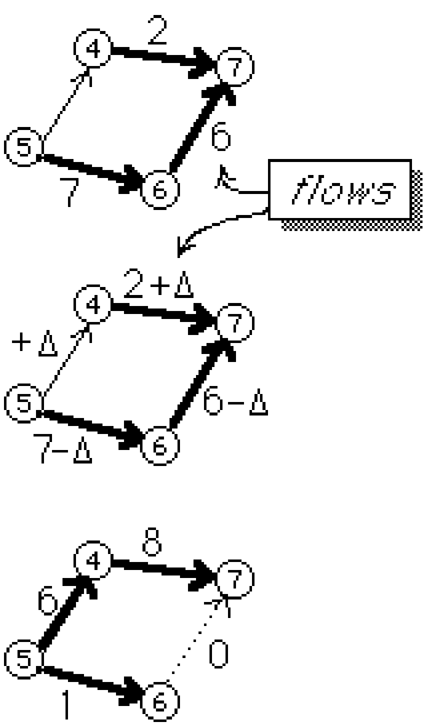


$\overline{C}_{43} = 1 - (5-5) = 1 \quad > 0$

$\overline{C}_{45} = 3 - (5-9) = 7 \quad > 0$

$\overline{C}_{54} = 2 - (9-5) = -2 < 0$

Negative! Arc (5,4) should enter basis



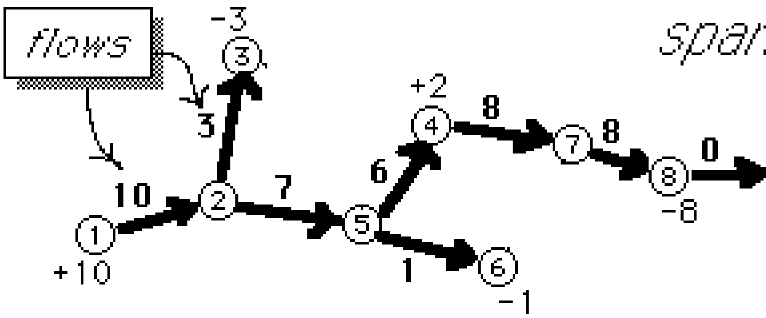
Adding arc (5,4) to the tree will create a cycle.

Increase flow in (5,4) by an increment Δ.

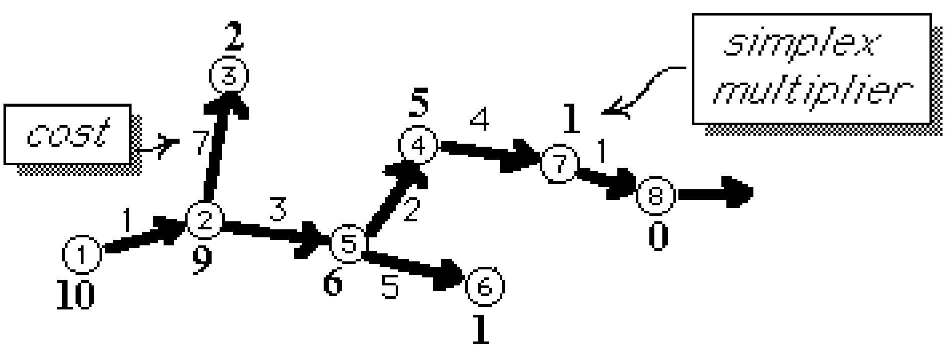
Adjust other flows around the cycle.

Maximum value for Δ *is 6, the minimum of flows being decreased. Arc (6,7) leaves the basis.*

The new basis (rooted spanning tree)

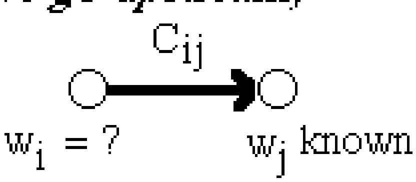


Thus, one simplex iteration is completed. The algorithm continues until no negative reduced cost remains.



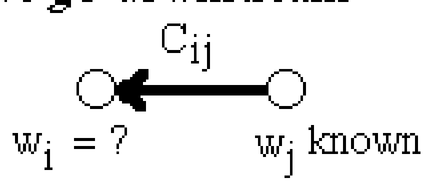
As we travel away from the root node toward the ends of the branches,

if we go upstream,



add cost: $\mathbf{w}_i = \mathbf{w}_j + \mathbf{c}_{ij}$

if we go downstream



subtract cost: $\mathbf{w}_i = \mathbf{w}_j - \mathbf{c}_{ij}$

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By using a variant of the simplex method known as the "upper bounding technique", it is possible to handle easily the more common network problem in which there are upper &/or lower bounds on the flows in the arcs.

In UBT (upper bounding technique), a nonbasic variable may be either at the lower ***or*** *upper bound.*