

Multiple-Facility Location in the Plane



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Given

a set of demand points (X_j, Y_j) , $j=1, 2, \dots, n$
with volume of demand w_j

Find

a set of m facility locations, and
an allocation of the demand points to
these facilities, so as to minimize
the sum of the transportation costs

Two aspects of decisions to be made:

Location

(\hat{X}^i, \hat{Y}^i) = location of facility # i

Allocation

J_i = set of demand points to be served
by facility #i
where

$$J_1 \cup J_2 \cup \dots \cup J_m = \{1, 2, \dots, n\}$$

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Objective is a function of both the location and allocation, i.e., of (\hat{X}^i, \hat{Y}^i) & J_i

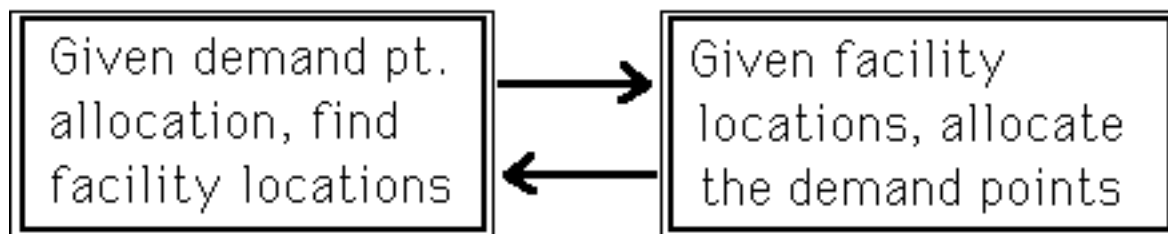
Total Transportation Cost

$$C(\hat{X}^1, \hat{Y}^1, J_1, \dots, \hat{X}^m, \hat{Y}^m, J_m) = \sum_{i=1}^m \sum_{j \in J_i} w_j \sqrt{(\hat{X}^i - X_j)^2 + (\hat{Y}^i - Y_j)^2}$$

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Optimizing simultaneously with respect to
 the facility locations
 the allocation of demand points to facilities
 is quite difficult.

One heuristic approach is to alternate between
 a location problem & allocation problem:



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Heuristic Algorithm

1. Partition the set of n demand points into m subsets $J_i, i=1, \dots, m, J_1 \cup \dots \cup J_m = \{1, 2, \dots, n\}$
2. *Location*: For each subset J_i , solve optimally a single-source facility location problem, to obtain facility locations

$$(\hat{X}^i, \hat{Y}^i), i=1, 2, \dots, m$$

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3. *Re-Allocation:* Allocate each demand point to the nearest facility location. That is,

$$\text{If } j \in J_{i'} \text{ but } d_j(\hat{X}^{i''}, \hat{Y}^{i''}) < d_j(\hat{X}^{i'}, \hat{Y}^{i'})$$

$$\text{then } J_{i''} = J_{i''} \cup \{j\}, J_{i'} = J_{i'} / \{j\}$$

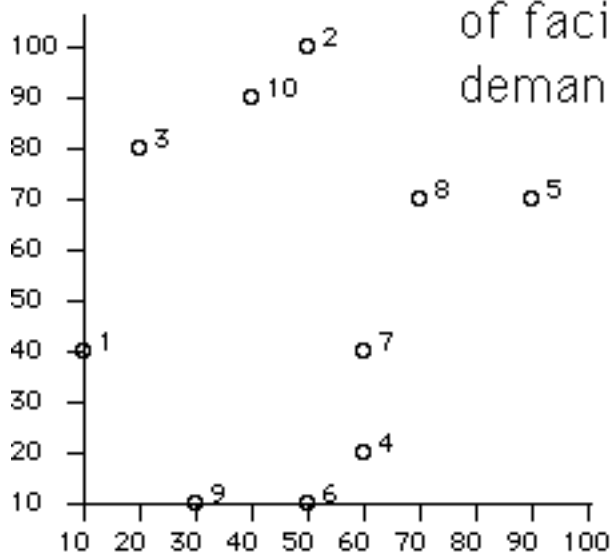
4. If no demand point was re-allocated in step 3, then STOP. Otherwise, return to step 2.

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Example

Cost of operating facility = \$75/day

Find the optimal number & locations of facilities to satisfy the daily demand:



| i | W_i |
|----|-------|
| 1 | 0.75 |
| 2 | 0.6 |
| 3 | 0.75 |
| 4 | 0.6 |
| 5 | 1.2 |
| 6 | 0.5 |
| 7 | 0.4 |
| 8 | 0.5 |
| 9 | 1.05 |
| 10 | 0.6 |

shipping cost
is
 W_i times
the distance
shipped

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Single-Source
Weber Problem

The optimal single source serving the demand points is located at:

$$X = 50.2611, Y = 51.711$$

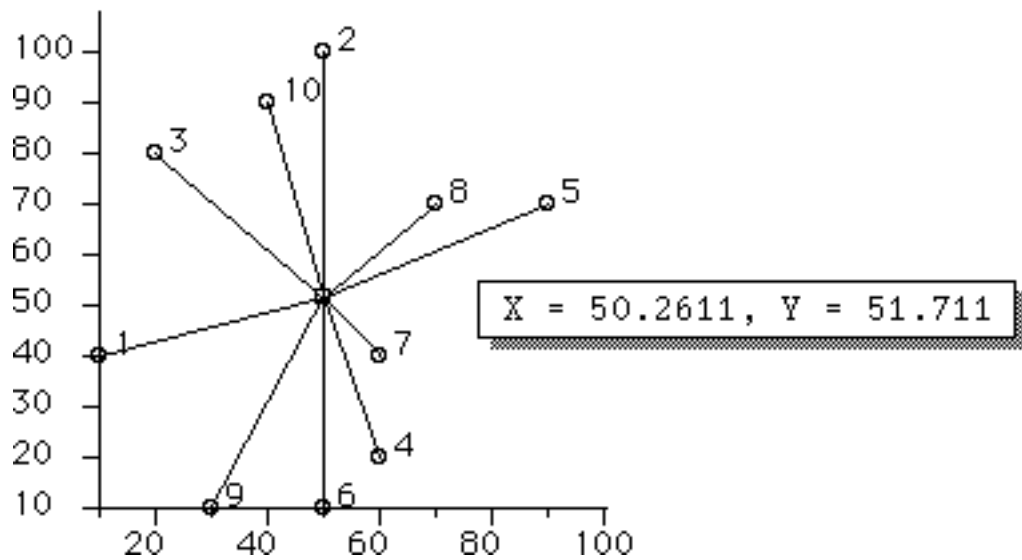
| i | $C_i d_i$ |
|----|-----------|
| 1 | 31.4479 |
| 2 | 28.964 |
| 3 | 31.058 |
| 4 | 19.9137 |
| 5 | 52.4907 |
| 6 | 20.864 |
| 7 | 6.09857 |
| 8 | 13.4506 |
| 9 | 48.7037 |
| 10 | 23.774 |

Total daily cost:
 $276.765 + 75$
 $= 351.765$

Total cost = 276.765
(shipping cost.)

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Optimal Single Facility Location



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Locating Two Facilities

Let's begin by partitioning the set of demand points into two subsets:

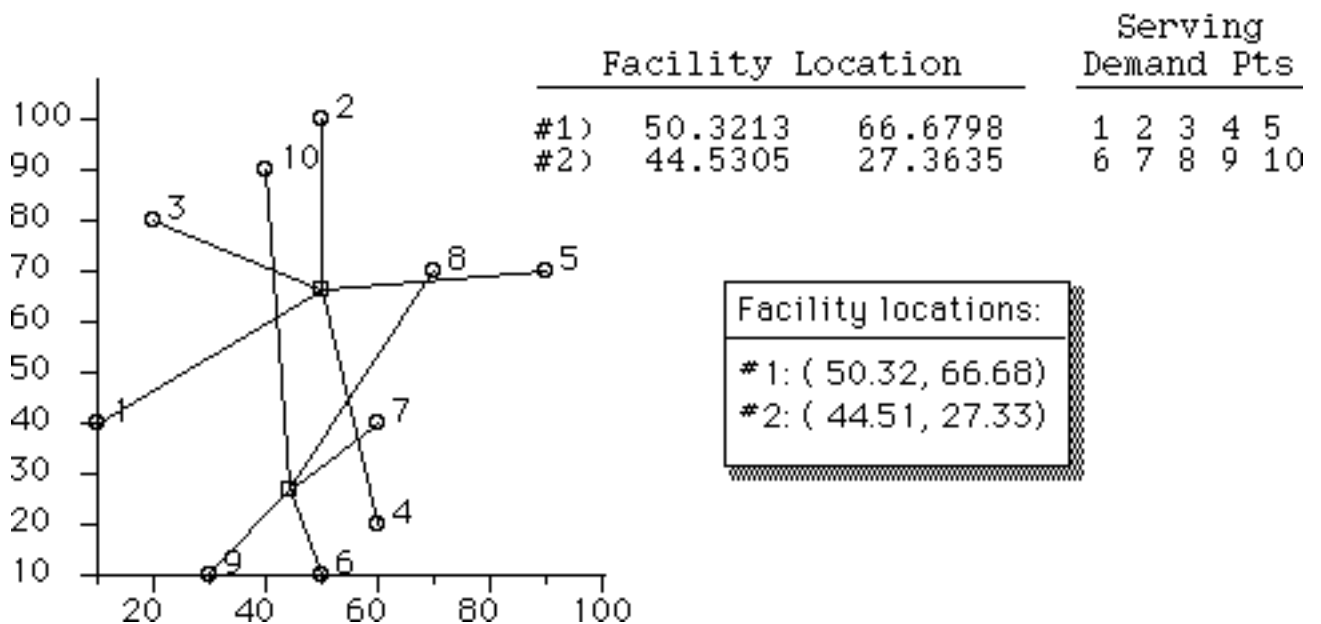
$$J_1 = \{1,2,3,4,5\}$$

$$J_2 = \{6,7,8,9,10\}$$

We then find the optimal location of a facility to serve the demand points J_1 , and another facility to serve the demand points J_2 .

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Iteration 1



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Allocation of Demand Points

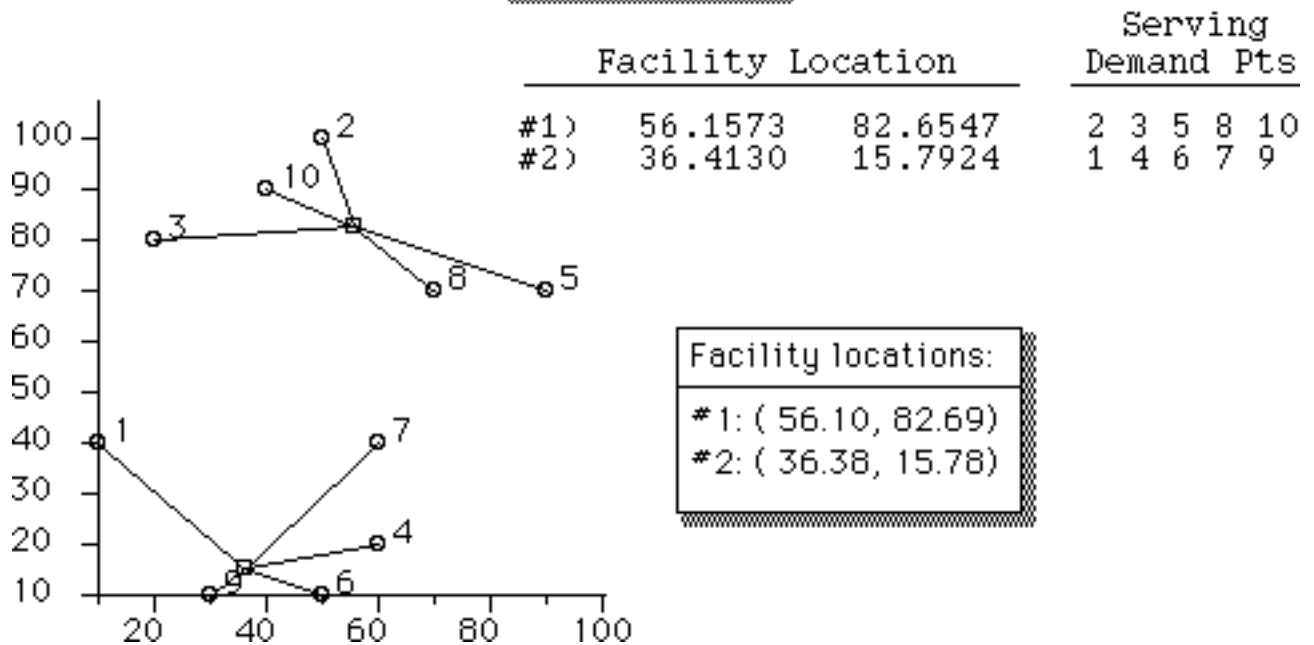
| i | Shipping Cost | Old Source | Source |
|----|---------------|------------|--------|
| 1 | 36.2617 | 27.5775 | 1 → 2 |
| 2 | 19.993 | 43.7053 | 1 → 1 |
| 3 | 24.8386 | 43.5539 | 1 → 1 |
| 4 | 28.6036 | 10.2796 | 1 → 2 |
| 5 | 47.7808 | 74.7991 | 1 → 1 |
| 6 | 28.3404 | 9.10228 | 2 → 2 |
| 7 | 11.3525 | 7.98987 | 2 → 2 |
| 8 | 9.97841 | 24.8323 | 2 → 1 |
| 9 | 63.2233 | 23.7733 | 2 → 2 |
| 10 | 15.3013 | 37.6801 | 2 → 1 |

Total shipping cost after re-allocating demand: 196.615

Total shipping cost before re-allocation: 260.856

(Improvement from re-allocation: 64.2409)

Iteration 2



| |
|-----------------------------|
| Allocation of Demand Points |
|-----------------------------|

| i | Shipping Cost | Old Source | Source |
|----|---------------|------------|--------|
| 1 | 47.1363 | 26.8711 | 2 |
| 2 | 11.0435 | 51.178 | 1 |
| 3 | 27.191 | 49.7041 | 1 |
| 4 | 37.6634 | 14.3756 | 2 |
| 5 | 43.3575 | 91.4683 | 1 |
| 6 | 36.4576 | 7.38512 | 2 |
| 7 | 17.131 | 13.5195 | 2 |
| 8 | 9.37765 | 31.8848 | 1 |
| 9 | 81.0808 | 9.07374 | 2 |
| 10 | 10.6491 | 44.5765 | 1 |

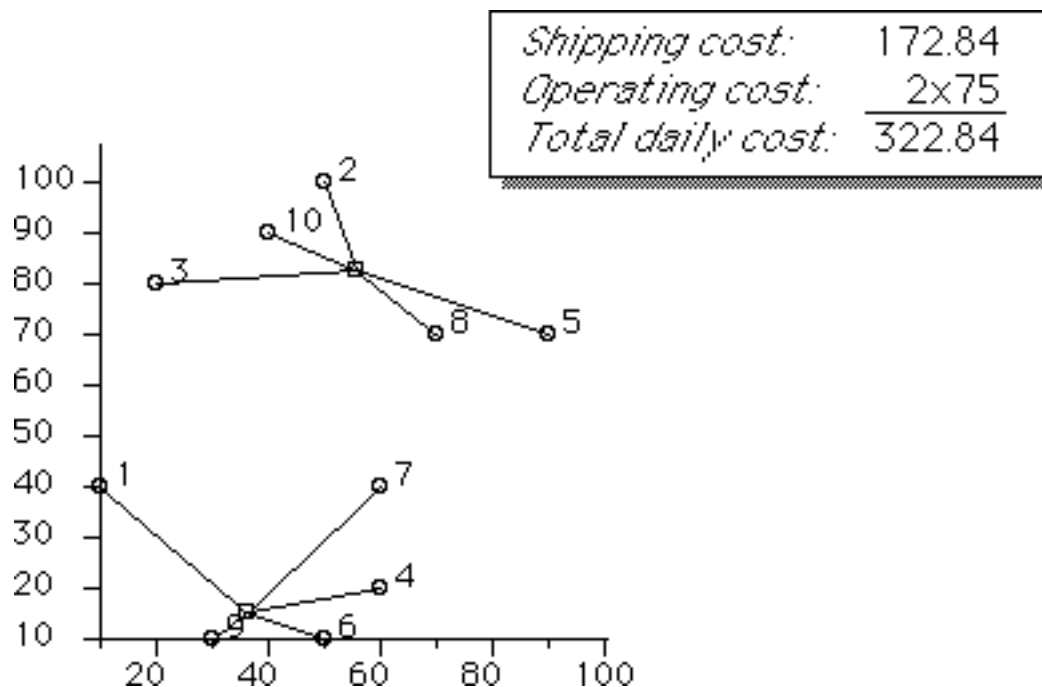
Total shipping cost after re-allocating demand: 172.844

Total shipping cost before re-allocation: 172.844

(Improvement from re-allocation: 0)

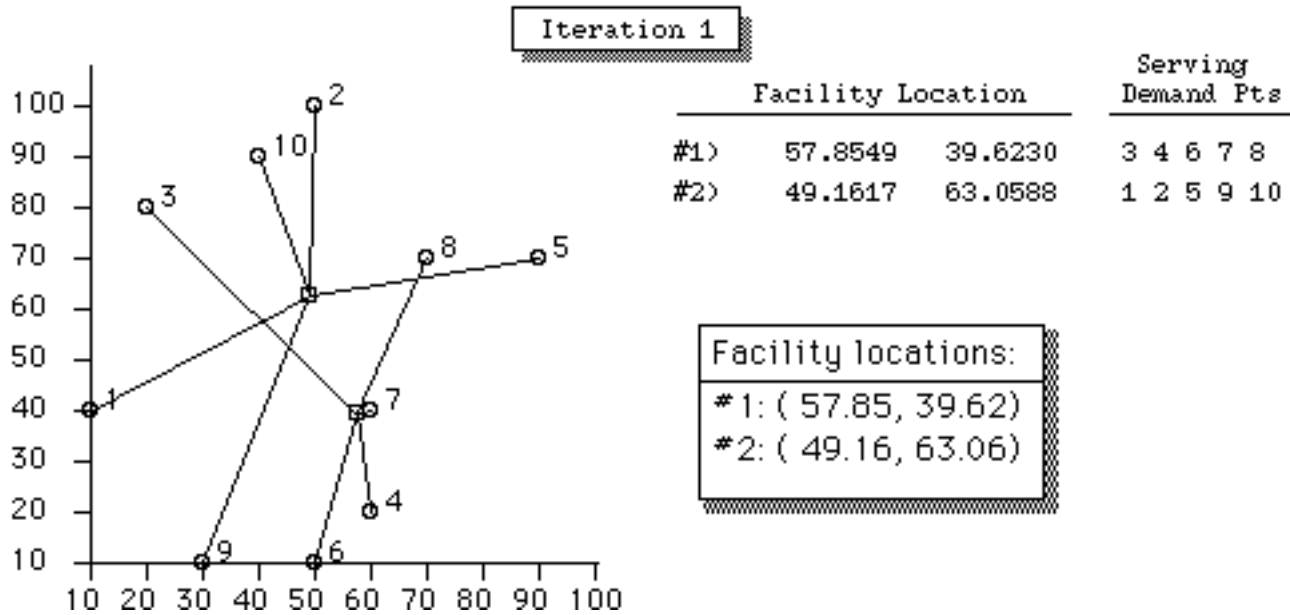
Converged!

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Starting from another initial (random) allocation:



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Allocation of Demand Points

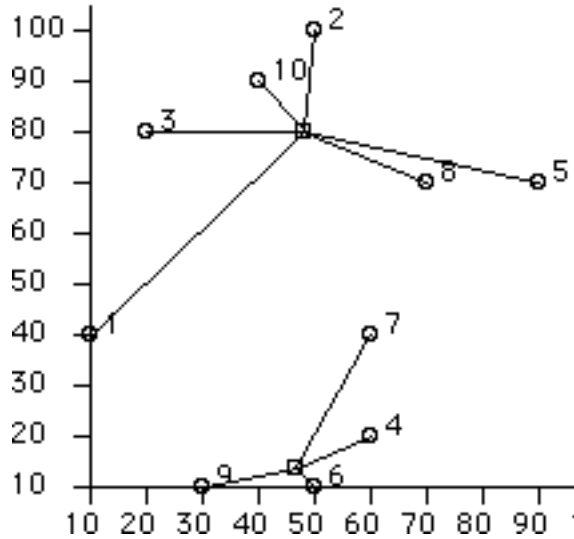
| i | Shipping Cost | | Old Source | Source |
|----|---------------|---------|------------|--------|
| 1 | 35.8923 | 34.0846 | 2 | 2 |
| 2 | 36.5315 | 22.1704 | 2 | 2 |
| 3 | 41.5103 | 25.2941 | 1 | 2 |
| 4 | 11.8439 | 26.6412 | 1 | 1 |
| 5 | 53.073 | 49.7088 | 2 | 2 |
| 6 | 15.3233 | 26.5327 | 1 | 1 |
| 7 | 0.87121 | 10.1916 | 1 | 1 |
| 8 | 16.3575 | 10.982 | 1 | 2 |
| 9 | 42.6953 | 59.2335 | 2 | 1 |
| 10 | 32.0685 | 17.0738 | 2 | 2 |

Total shipping cost after re-allocating demand: 230.047
 Total shipping cost before re-allocation: 268.177
 (Improvement from re-allocation: 38.1299)

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Iteration 2

| | Facility Location | | Serving Demand Pts |
|-----|-------------------|---------|--------------------|
| #1) | 47.0367 | 13.7634 | 4 6 7 9 |
| #2) | 48.5206 | 80.2241 | 1 2 3 5 8 10 |



Facility locations:
 #1: (47.04, 13.76)
 #2: (48.52, 80.22)

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Allocation of Demand Points

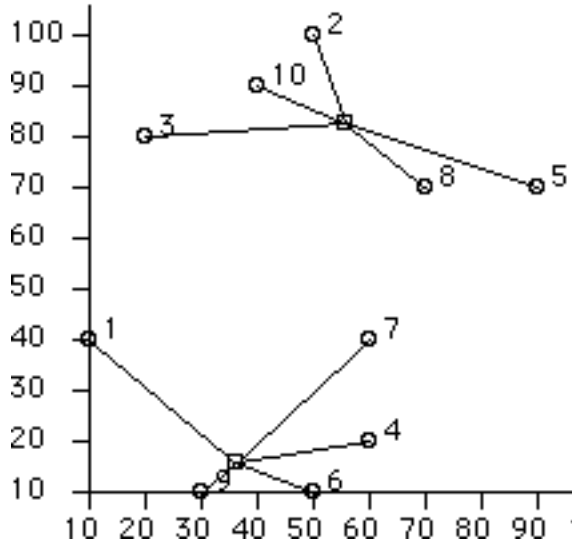
| i | Shipping Cost | Old Source | Source |
|----|---------------|------------|--------|
| 1 | 34.041 | 41.7704 | 2 1 |
| 2 | 51.7725 | 11.8987 | 2 2 |
| 3 | 53.6565 | 21.3911 | 2 2 |
| 4 | 8.63128 | 36.785 | 1 1 |
| 5 | 84.924 | 51.2651 | 2 2 |
| 6 | 2.39502 | 35.1198 | 1 1 |
| 7 | 11.7058 | 16.732 | 1 1 |
| 8 | 30.3721 | 11.8943 | 2 2 |
| 9 | 18.3198 | 76.2565 | 1 1 |
| 10 | 45.9364 | 7.7808 | 2 2 |

Total shipping cost after re-allocating demand: 179.323
 Total shipping cost before re-allocation: 187.052
 (Improvement from re-allocation: 7.72937)

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Iteration 3

| | Facility Location | | Serving Demand Pts |
|-----|-------------------|---------|--------------------|
| #1) | 36.4130 | 15.7924 | 1 4 6 7 9 |
| #2) | 56.1573 | 82.6547 | 2 3 5 8 10 |



Facility locations:

- # 1: (36.41, 15.79)
- # 2: (56.16, 82.65)

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Allocation of Demand Points

| i | Shipping Cost | Old Source | Source |
|----|---------------|------------|--------|
| 1 | 26.8711 | 47.1363 | 1 |
| 2 | 51.178 | 11.0435 | 2 |
| 3 | 49.7041 | 27.191 | 2 |
| 4 | 14.3756 | 37.6634 | 1 |
| 5 | 91.4683 | 43.3575 | 2 |
| 6 | 7.38512 | 36.4576 | 1 |
| 7 | 13.5195 | 17.131 | 1 |
| 8 | 31.8848 | 9.37765 | 2 |
| 9 | 9.07374 | 81.0808 | 1 |
| 10 | 44.5765 | 10.6491 | 2 |

Total shipping cost after re-allocating demand: 172.844

Total shipping cost before re-allocation: 172.844

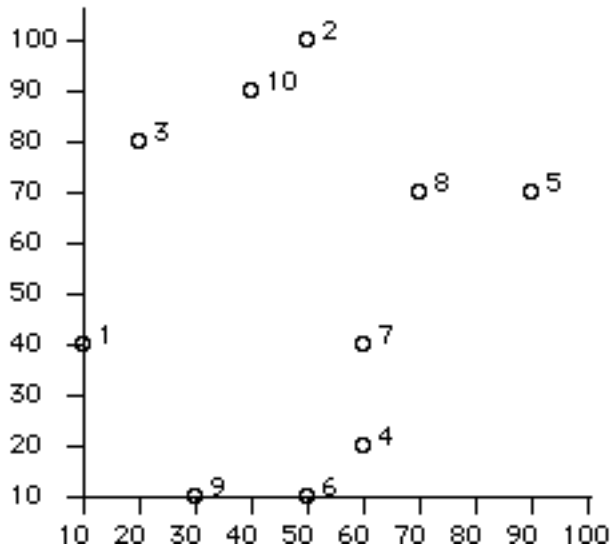
(Improvement from re-allocation: 0)

Converged!

(same as earlier solution!)

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Next we will find a solution with THREE facility locations.



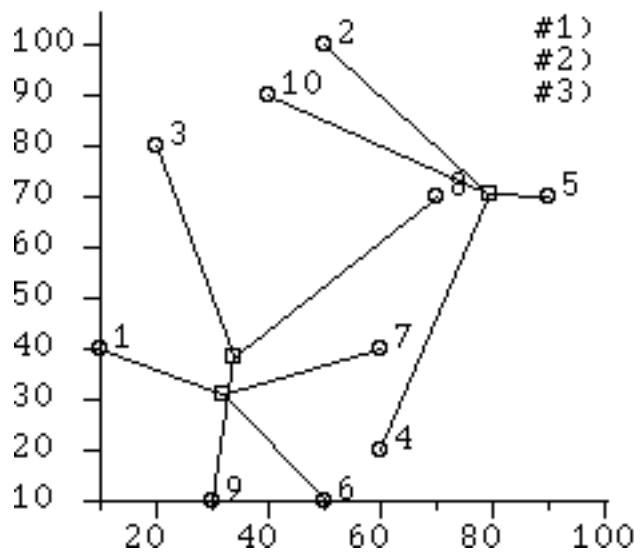
Random initial allocation:

| Demand Pts | | | |
|------------|---|---|----|
| 3 | 8 | 9 | |
| 1 | 6 | 7 | |
| 2 | 4 | 5 | 10 |

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Iteration 1

| | Facility Location | | Serving Demand Pts |
|-----|-------------------|---------|--------------------|
| #1) | 33.8389 | 38.5597 | 3 8 9 |
| #2) | 32.3758 | 31.4353 | 1 6 7 |
| #3) | 79.9591 | 70.9838 | 2 4 5 10 |



| Facility locations: | |
|---------------------|-----------------|
| #1: | (33.84, 38.56) |
| #2: | (32.38, 31.44) |
| #3: | (79.96, 70.98) |

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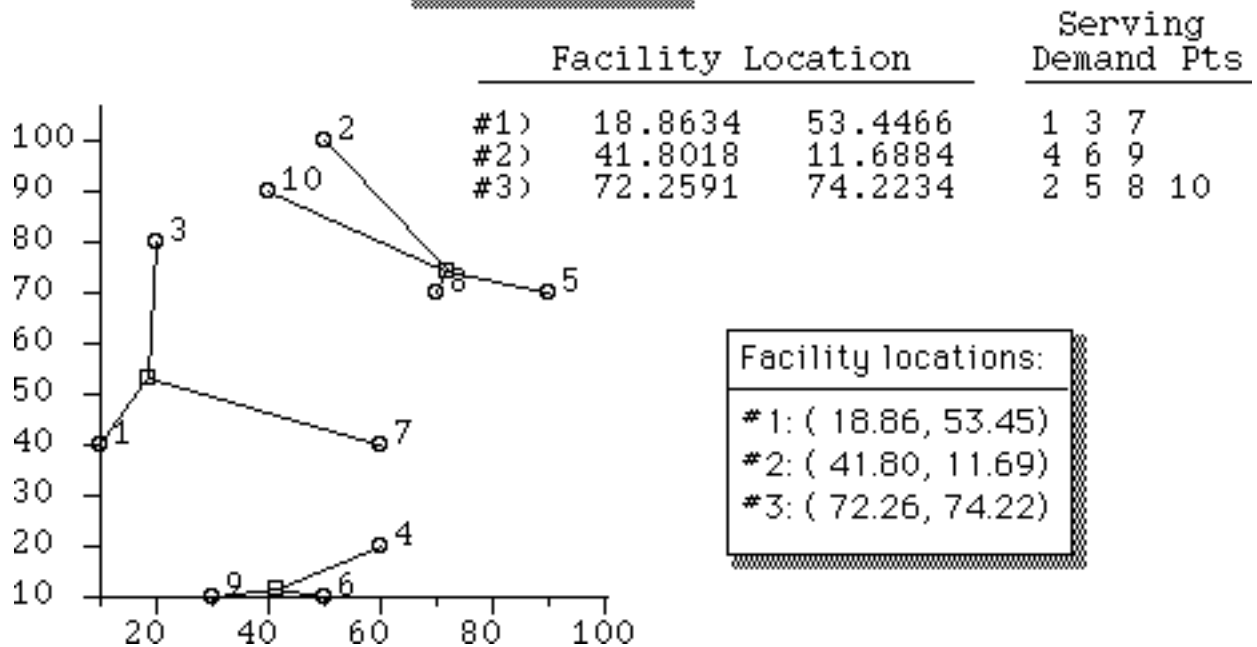
Allocation of Demand Points

| i | Shipping Cost | | | Old Source | Source |
|----|---------------|---------|---------|------------|--------|
| 1 | 17.9118 | 17.9692 | 57.3849 | 2 | → 1 |
| 2 | 38.1181 | 42.4761 | 25.0243 | 3 | → 3 |
| 3 | 32.7675 | 37.5876 | 45.4749 | 1 | → 1 |
| 4 | 19.2456 | 17.9385 | 32.8508 | 3 | → 2 |
| 5 | 77.2353 | 83.2058 | 12.1068 | 3 | → 3 |
| 6 | 16.4076 | 13.8752 | 33.9727 | 2 | → 2 |
| 7 | 10.4803 | 11.5686 | 14.7424 | 2 | → 1 |
| 8 | 23.9589 | 26.9389 | 5.00379 | 1 | → 3 |
| 9 | 30.2574 | 22.6449 | 82.7766 | 1 | → 2 |
| 10 | 31.0848 | 35.4353 | 26.5519 | 3 | → 3 |

Total shipping cost after re-allocating demand: 184.305
 Total shipping cost before re-allocation: 226.931
 (Improvement from re-allocation: 42.6256)

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Iteration 2



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Allocation of Demand Points

| i | Shipping Cost | | | Old Source | Source |
|----|---------------|---------|---------|------------|--------|
| 1 | 12.0787 | 31.9336 | 53.284 | 1 | 1 |
| 2 | 33.6038 | 53.2148 | 20.4344 | 3 | 3 |
| 3 | 19.9333 | 53.7797 | 39.433 | 1 | 1 |
| 4 | 31.8107 | 12.0038 | 33.3552 | 2 | 2 |
| 5 | 87.6446 | 90.7831 | 21.8841 | 3 | 3 |
| 6 | 26.7259 | 4.18512 | 33.9857 | 2 | 2 |
| 7 | 17.3114 | 13.4624 | 14.5411 | 1 | 2 |
| 8 | 26.8745 | 32.3859 | 2.39481 | 3 | 3 |
| 9 | 47.0937 | 12.5181 | 80.7236 | 2 | 2 |
| 10 | 25.3347 | 46.9994 | 21.5462 | 3 | 3 |

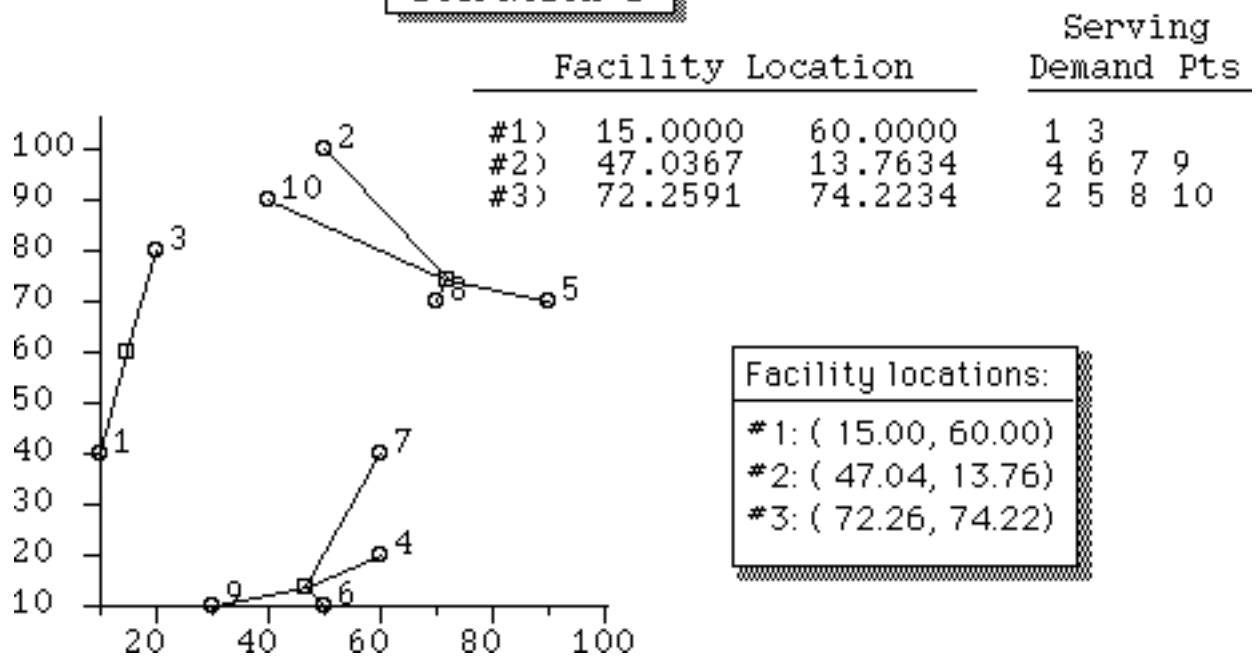
Total shipping cost after re-allocating demand: 140.441

Total shipping cost before re-allocation: 144.29

(Improvement from re-allocation: 3.84903)

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Iteration 2



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| |
|-----------------------------|
| Allocation of Demand Points |
|-----------------------------|

| i | Shipping Cost | Old Source | Source | | |
|----|---------------|------------|---------|---|---|
| 1 | 15.4616 | 34.041 | 53.284 | 1 | 1 |
| 2 | 31.8904 | 51.7725 | 20.4344 | 3 | 3 |
| 3 | 15.4616 | 53.6565 | 39.433 | 1 | 1 |
| 4 | 36.1248 | 8.63128 | 33.3552 | 2 | 2 |
| 5 | 90.7965 | 84.924 | 21.8841 | 3 | 3 |
| 6 | 30.5164 | 2.39502 | 33.9857 | 2 | 2 |
| 7 | 19.6977 | 11.7058 | 14.5411 | 2 | 2 |
| 8 | 27.9508 | 30.3721 | 2.39481 | 3 | 3 |
| 9 | 54.8116 | 18.3198 | 80.7236 | 2 | 2 |
| 10 | 23.4307 | 45.9364 | 21.5462 | 3 | 3 |

Total shipping cost after re-allocating demand: 138.235

Total shipping cost before re-allocation: 138.235

(Improvement from re-allocation: 0)

Converged!

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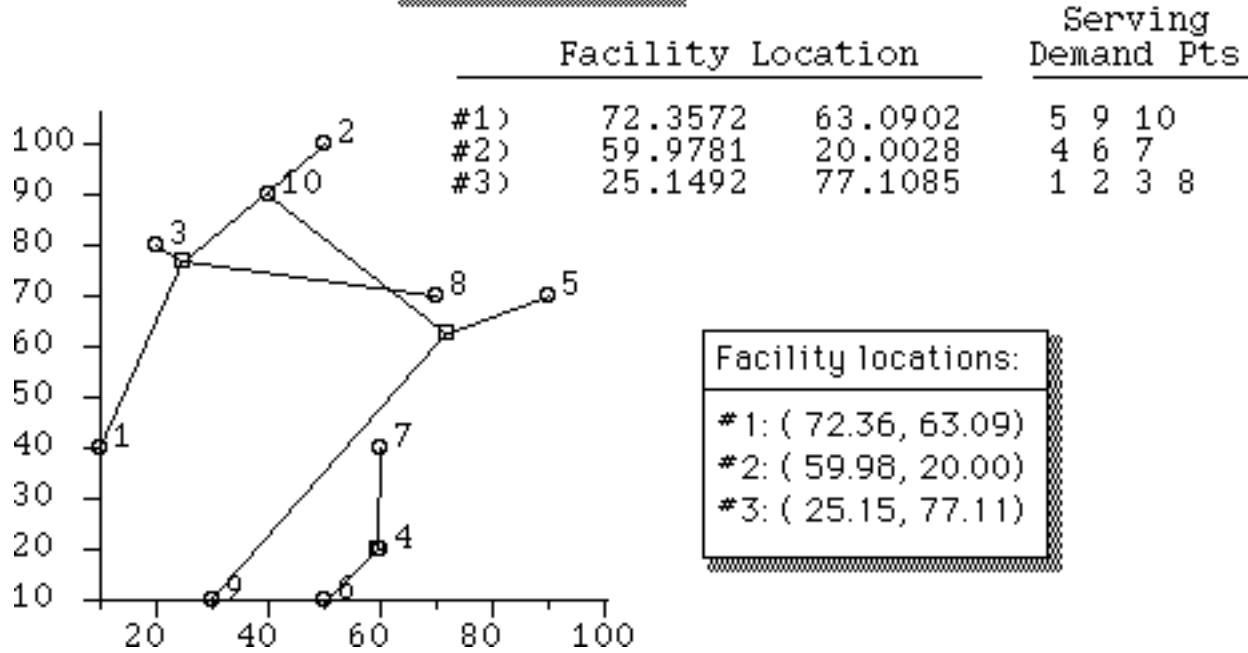
Starting from a different (random) allocation of the demand points:

| Demand Pts |
|------------|
| 5 9 10 |
| 4 6 7 |
| 1 2 3 8 |

Will the algorithm converge to the same facility locations?

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Iteration 1



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Allocation of Demand Points

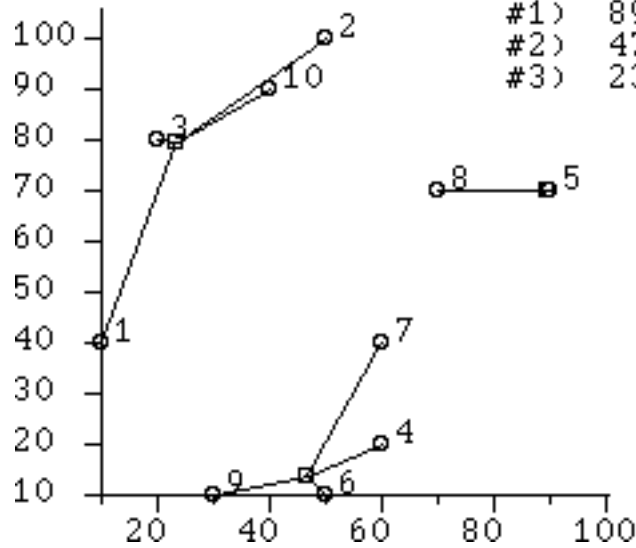
| i | Shipping Cost | Old Source | Source | | |
|----|---------------|------------|---------|---|---|
| 1 | 49.8712 | 40.3727 | 30.0612 | 3 | 3 |
| 2 | 25.8918 | 48.3702 | 20.2724 | 3 | 3 |
| 3 | 41.2651 | 54.0724 | 4.42914 | 3 | 3 |
| 4 | 26.8963 | 0.0132355 | 40.1416 | 2 | 2 |
| 5 | 22.7372 | 69.982 | 78.2871 | 1 | 1 |
| 6 | 28.8028 | 7.06434 | 35.781 | 2 | 2 |
| 7 | 10.4756 | 7.99887 | 20.3632 | 2 | 2 |
| 8 | 3.65039 | 25.4959 | 22.7053 | 3 | 1 |
| 9 | 71.3127 | 33.1831 | 70.6478 | 1 | 2 |
| 10 | 25.2508 | 43.6754 | 11.7994 | 1 | 3 |

Total shipping cost after re-allocating demand: 141.209
 Total shipping cost before re-allocation: 211.845
 (Improvement from re-allocation: 70.636)

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Iteration 2

| | Facility Location | | Serving Demand Pts | | | |
|-----|-------------------|---------|--------------------|---|---|----|
| #1) | 89.9924 | 70.0000 | 5 | 8 | | |
| #2) | 47.0367 | 13.7634 | 4 | 6 | 7 | 9 |
| #3) | 23.6396 | 79.8604 | 1 | 2 | 3 | 10 |



| Facility locations: | |
|---------------------|-----------------|
| #1: | (89.99, 70.00) |
| #2: | (47.04, 13.76) |
| #3: | (23.64, 79.86) |

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Allocation of Demand Points

| i | Shipping Cost | | | Old Source | Source |
|----|---------------|---------|---------|------------|--------|
| 1 | 64.0747 | 34.041 | 31.5971 | 3 | 3 |
| 2 | 29.9964 | 51.7725 | 19.904 | 3 | 3 |
| 3 | 53.0274 | 53.6565 | 2.7317 | 3 | 3 |
| 4 | 34.9834 | 8.63128 | 42.0229 | 2 | 2 |
| 5 | 0.00908125 | 84.924 | 80.5068 | 1 | 1 |
| 6 | 36.0534 | 2.39502 | 37.3341 | 2 | 2 |
| 7 | 16.9684 | 11.7058 | 21.5812 | 2 | 2 |
| 8 | 9.99622 | 30.3721 | 23.6987 | 1 | 1 |
| 9 | 89.0898 | 18.3198 | 73.6568 | 2 | 2 |
| 10 | 32.3068 | 45.9364 | 11.5486 | 3 | 3 |

Total shipping cost after re-allocating demand: 116.839

Total shipping cost before re-allocation: 116.839

(Improvement from re-allocation: 0)

Converged!

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| Facility Location | | Serving Demand Pts |
|----------------------|-----------------|--------------------|
| #1) | 15.0000 60.0000 | 1 3 |
| #2) | 47.0367 13.7634 | 4 6 7 9 |
| #3) | 72.2591 74.2234 | 2 5 8 10 |
| Total Shipping Cost: | | 138.235 |

| Facility Location | | Serving Demand Pts |
|----------------------|-----------------|--------------------|
| #1) | 89.9924 70.0000 | 5 8 |
| #2) | 47.0367 13.7634 | 4 6 7 9 |
| #3) | 23.6396 79.8604 | 1 2 3 10 |
| Total Shipping Cost: | | 116.839 |

Starting from two different initial allocations of the demand points, we obtain two different final solutions!

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Using 3 facilities:

Shipping cost: 116.84

Operating cost: 3×75.00

Total daily cost: 341.84

It appears that the optimal solution is to use 2 facilities:

| | |
|--------------------------|---------------|
| <i>Shipping cost:</i> | 172.84 |
| <i>Operating cost:</i> | 2×75 |
| <i>Total daily cost:</i> | 322.84 |



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